The Chow-Lam Form

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With Bernd Sturmfels (The Chow-Lam Form)
and Kristian Ranestad (Chow-Lam Recovery)

Linear projections

Consider a linear projection $Z : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{r-1}$.

Q: Fix a variety $\mathcal{V} \subset \mathbb{P}^{n-1}$ of dimension $r-2$. How to recover $\mathcal{V}$ from ALL linear projections such that $Z(\mathcal{V})$ is a hypersurface?

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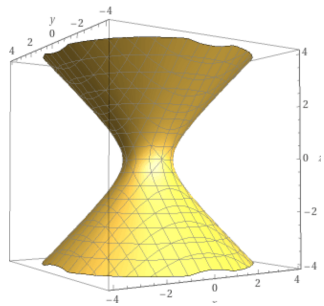
Not equality! We call the right side the *recovered variety* $W_{\mathcal{V}}$.

Example: curves

Let $\mathcal{V} \subset \mathrm{Gr}(2,n)$ be a curve. Consider the ruled surface

$$X_{\mathcal{V}} := \overline{\bigcup_{[L]\in\mathcal{V}} L} \subset \mathbb{P}^{n-1}.$$



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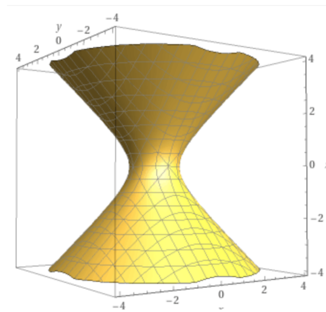

Theorem (P-Ranestad 25)

Let $\mathcal{V}, \mathcal{V}'$ be curves in Gr$(2,n)$. Suppose that $X_{\mathcal{V}}$ and $X_{\mathcal{V}'}$ are not cones. Then $W_{\mathcal{V}} = W_{\mathcal{V}'}$ if and only if $X_{\mathcal{V}} = X_{\mathcal{V}'}$.

Example

If X is a quadric surface in $\mathbb{P}^3$, it has two rulings $\mathcal{V}, \mathcal{V}'$. Then $W_{\mathcal{V}} = W_{\mathcal{V}'} = \mathcal{V} \cup \mathcal{V}'$. These curves have all the same $\wedge^k Z$-projections, where Z is of dimension $4 \times n$.

The amplituhedron

The *positive Grassmannian* is

$$\mathsf{Gr}^{\geq 0}(k,n) := \mathsf{Gr}(k,n) \cap \mathbb{P}_{\geq 0}^{\binom{n}{k}-1}.$$

The *amplituhedron* $\mathcal{A}_{k,m,n}(Z)$ is the image of

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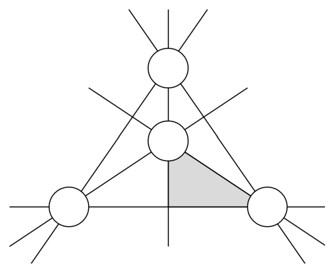
- Computes scattering amplitudes in $N = 4$ super Yang-Mills
- Inspired *positive geometry*, which studies "positive" parts of varieties, e.g. $\mathsf{Gr}^{\geq 0}(k,n)$, $\mathsf{Fl}^{\geq 0}(n)$, $\mathcal{M}_{0,n}^{\geq 0}$ ...



Combinatorial & Computational Aspects of Positive Geometry (Fri)

The Chow-Lam form (P-Sturmfels 25)

Fix a variety $\mathcal{V} \subset \mathrm{Gr}(k, n)$ of dimension $k(r-k)-1$. The *Chow-Lam locus* is

$$\mathcal{CL}_{\mathcal{V}} := \{P \in \mathrm{Gr}(k{+}n{-}r, n) : \exists Q \in \mathcal{V} \text{ with } Q \text{ is a subspace of } P\}.$$

The *Chow-Lam form* $CL_{\mathcal{V}}$ is its equation, or $CL_{\mathcal{V}} := 1$ if it is not a hypersurface. If $k = 1$, then we call it the *Chow form*.

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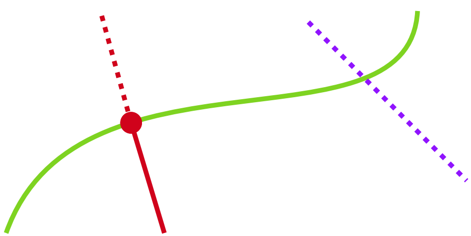
Example (Cayley 1860)

Let $k=1, n=4, r=3$. Then $\mathcal{V}$ is a curve in $\mathbb{P}^3$ and $\mathcal{CL}_{\mathcal{V}}$ is a hypersurface in $\mathrm{Gr}(2,4)$. It consists of lines in $\mathbb{P}^3$ meeting $\mathcal{V}$.



Discriminant!

Lines meeting the twisted cubic

Consider the closure of

$$t \mapsto [1 : t : t^2 : t^3] \ \in \ \mathbb{P}^3.$$

The Chow form is the determinant of the *Bézout matrix*:

$$\mathrm{CL}_{\mathcal{V}} = \det \begin{bmatrix} p_{12} & p_{13} & p_{14} \\ p_{13} & p_{14} + p_{23} & p_{24} \\ p_{14} & p_{24} & p_{34} \end{bmatrix}.$$

Its expansion is

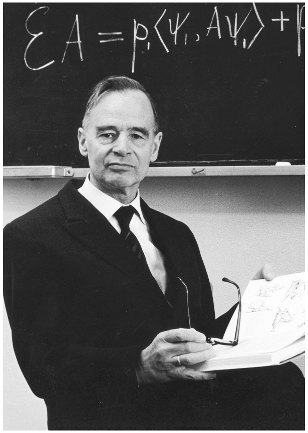
$$-p_{14}^3 - p_{14}^2 p_{23} + 2p_{13}p_{14}p_{24} - p_{12}p_{24}^2 - p_{13}^2 p_{34} + p_{12}p_{14}p_{34} + p_{12}p_{23}p_{34}.$$

If p_{ij} are maximal minors of $\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{bmatrix}$, then

$$\mathrm{CL}_{\mathcal{V}} = 0 \quad \Longleftrightarrow \quad \begin{cases} a_3t^3 + a_2t^2 + a_1t + a_0 \\ b_3t^3 + b_2t^2 + b_1t + b_0 \end{cases} \text{share a root.}$$

History

- Curves in $\mathbb{P}^3$ due to Cayley in 1860
- Projective varieties (so $k = 1$) due to Chow and van der Waerden in 1937 $\rightsquigarrow$ Chow form



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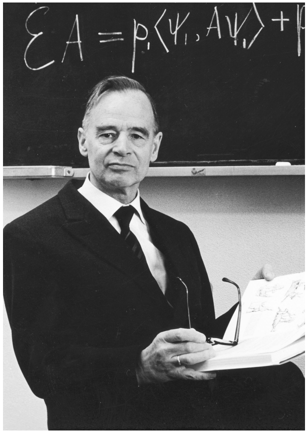

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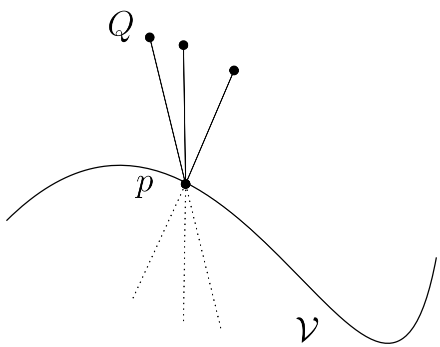
Special properties for $k = 1$:

- ⋄ Always a hypersurface of the same degree as $\mathcal{V}$
- ⋄ Can recover equations of $\mathcal{V}$ from $\mathrm{CL}_{\mathcal{V}}$

When can you recover $\mathcal{V}$ from $\mathrm{CL}_{\mathcal{V}}$?

Example (Curve in $\mathbb{P}^3$)

Suppose $\mathcal{V}$ is a curve and we know $\mathcal{CL}_{\mathcal{V}}$. Then p is in $\mathcal{V} \iff$ every line Q containing p is in $\mathcal{CL}_{\mathcal{V}}$.



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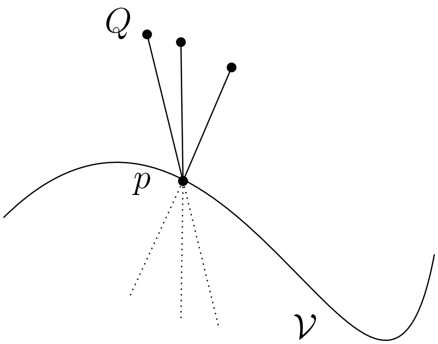

The *Chow-Lam recovered variety* of $\mathcal{V}$ is

$$W'_{\mathcal{V}} := \{P \in Gr(k,n) : \text{ every } Q \text{ containing } P \text{ is in } \mathcal{CL}_{\mathcal{V}}\}.$$

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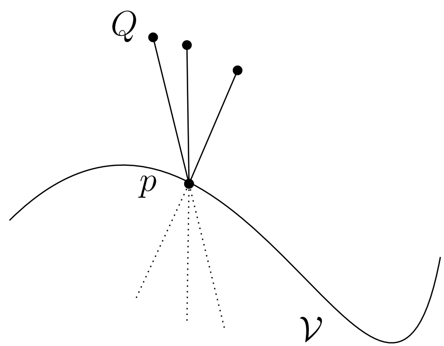

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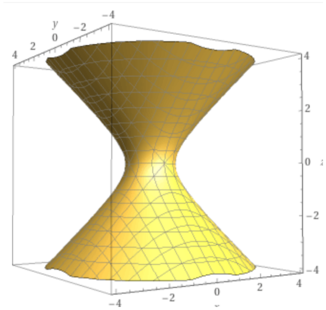
Theorem (P-Ranestad 25)

We have the equality

$$W'_{\mathcal{V}} = W_{\mathcal{V}} := \bigcap_Z (\wedge^k Z)^{-1}(\wedge^k Z(\mathcal{V})).$$

A curve in Gr$(2, 4)$

Let $\mathcal{V}$ be a curve in Gr$(2, 4)$. If $X_{\mathcal{V}}^{\vee}$ is a surface, then $\mathcal{CL}_{\mathcal{V}} = X_{\mathcal{V}}^{\vee}$.



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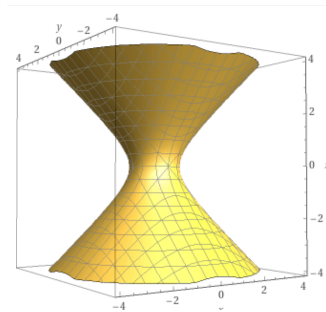

Corollary

Let $\mathcal{V}, \mathcal{V}'$ be curves in Gr$(2,4)$. If $X_{\mathcal{V}} = X_{\mathcal{V}'}$ and both are surfaces, then $\mathcal{CL}_{\mathcal{V}} = \mathcal{CL}_{\mathcal{V}'}$ and $W_{\mathcal{V}} = W_{\mathcal{V}'}$.

A Schubert example

Let $\mathcal{V} \subset \mathrm{Gr}(2,5)$ consist of all lines meeting three fixed planes P_1, P_2, and P_3 in $\mathbb{P}^4$. Then $W_{\mathcal{V}}$ equals

$$\{\text{lines } L \text{ in } \mathbb{P}^4 : \text{every plane } Q \text{ containing } L \text{ also contains some } L' \text{ in } \mathcal{V}\}.$$

Then $W_{\mathcal{V}}$ has 7 components, including

1. Lines contained in P_i
2. Lines meeting $P_i \cap P_j$

Example: Suppose L is contained in P_1.

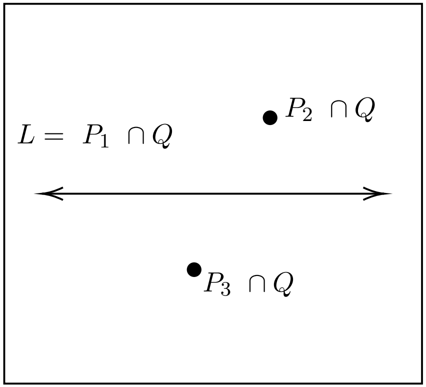

Figure 1: The geometry in Q containing L

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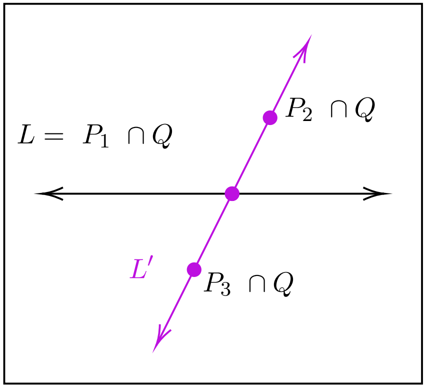


Figure 1: The geometry in Q containing L

Schubert varieties

Fix projective linear spaces $A \subset B$ of codimensions $a+1$ and $b+1$ in $\mathbb{P}^{n-1}$. We define

$$\Omega_{a,b}(A,B) = \{L \,:\, L \cap A \neq 0,\ L \subset B\} \ \subset \ \mathsf{Gr}(2,n).$$

If P has codimension 2 in $\mathbb{P}^{n-1}$, then

$$\Omega_1(A) := \{L \,:\, L \cap A \neq \varnothing\} \ \subset \ \mathsf{Gr}(2,n).$$

Example

The extra components in $W_{\mathcal{V}}$ for $\mathcal{V} = \Omega_1(P_1) \cap \Omega_1(P_2) \cap \Omega_1(P_3)$ are of types

- $\Omega_{1,1}$ (lines contained in P_i)
- Ω_2 (lines meeting $P_i \cap P_j$)

Some data for Schubert hyperplane sections

Consider the linear section

$$\Omega_1^r := \Omega_1(A_1) \cap \ldots \cap \Omega_1(A_r)$$

for $A_1, \ldots, A_r$ general.

$\mathcal{V}$	Recovered	n at least
Ω_1^7	Ω_7, Ω_{6^2}	9
Ω_1^9	Ω_9, Ω_{8^2}	11
Ω_1^{2i+1}	$\Omega_{2i+1}, \Omega_{(2i)^2}$	$2i+3$

Table 1: Some recovered components for $k = 2$

Theorem (P-Ranestad 25)

Fix k and $i > k$. Consider $V := \Omega_1^{ki+1} \subset Gr(k,n)$ for $n > k(i+1)+1$. Then the Chow-Lam recovery $W_{\mathcal{V}}$ will contain recovered components of Schubert types
$\Omega_{ki+1}, \Omega_{(k(i-1)+2)^2}, \Omega_{(k(i-2)+3)^3}, \ldots, \Omega_{(k(i-k+1)+k)^k}$.

Thank you for listening!

Curves in Gr$(2, n)$

Theorem (P-Ranestad 25)

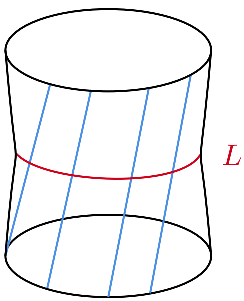
Let $\mathcal{V} \subset$ Gr$(2, n)$ be a curve such that $X_{\mathcal{V}}$ is not a cone. Then a line $L \in \mathbb{P}^{n-1}$ is in the recovery $W_{\mathcal{V}}$ if and only if $L \subset X_{\mathcal{V}}$.

Example (Hirzebruch surface)

Consider

$$X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(a)) \to \mathbb{P}^1.$$

Then X is ruled by the fibers, all of which meet the line L corresponding to $\mathcal{O}_{\mathbb{P}^1}(1)$.



We can embed $X \hookrightarrow \mathbb{P}^{a+2}$ using sections of the bundles. The ruling gives us a curve $\mathcal{V}$ in Gr$(2, a+3)$ with $L \in W_{\mathcal{V}}$.