D-modules Techniques for Feynman Integrals

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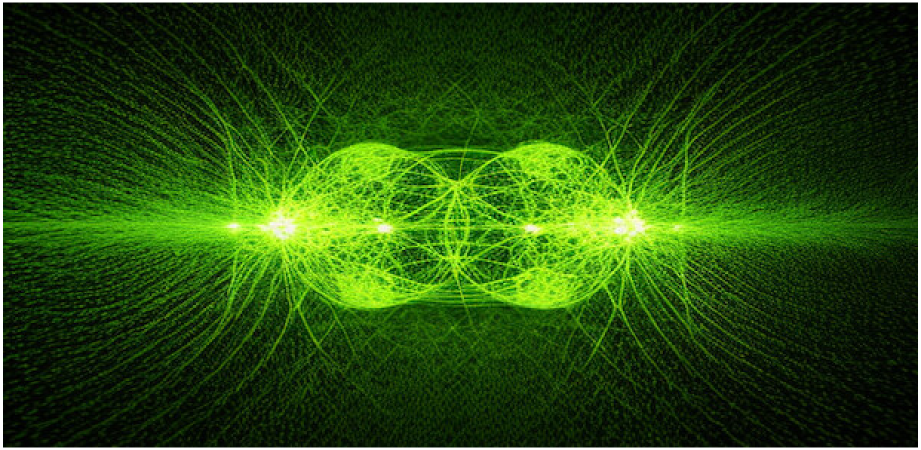
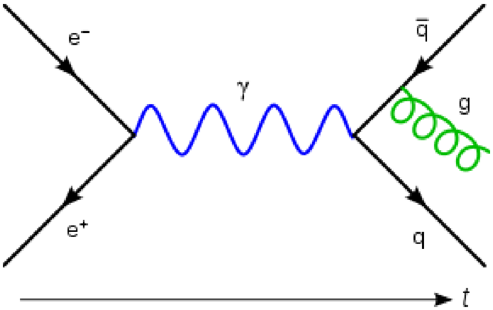


Overview

Scattering amplitudes to PDEs

D-modules: why are they useful?



This presentation is based on joint work with Johannes Henn, Anna-Laura Sattelberger, and Simone Zoia, available at https://arxiv.org/abs/2303.11105.

Observation 1: A scattering amplitude is a sum over Feynman integrals

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Scattering Amplitudes to PDEs

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Observation 2: Integration is difficult.

Idea: What if we exploited symmetry to find some differential equations that the integrals satisfy, and solve those?

Encoding Symmetry Using PDEs

Example (Dilation in two variables)

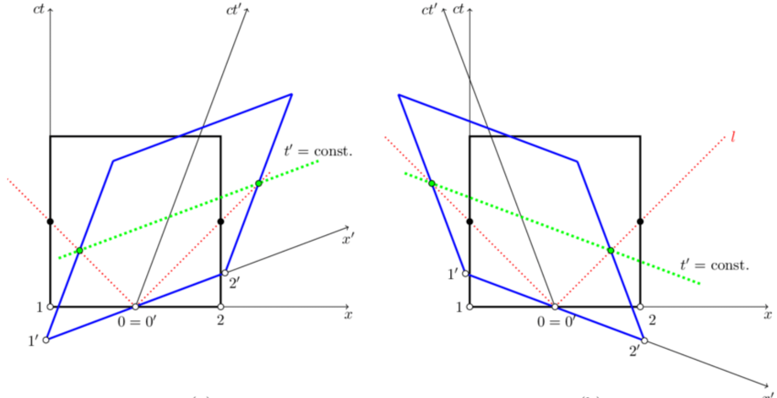
Observe that by Taylor expansion,

$$f((1 + \ )x; (1 + \ )y) = f(x; y) + \quad x\frac{df}{dx} + y\frac{df}{dy} \quad + O(\ ^2):$$

Thus $f(x; y)$ is invariant under infinitesimal dilation whenever $T(x) := x\frac{d}{dx} + y\frac{d}{dy}$ annihilates $f(x; y):$ Solutions are $\frac{x}{y};$ etc.

The scattering amplitude will be annihilated by these differential operators.

Conformal Symmetry



(a) (b)

Functions of these particles which are physically meaningful should be invariant under *conformal* (or angle-preserving) transformations.

PDEs for Conformal Symmetry

After translating from position to momentum space, the full list of operators that capture conformal symmetry is

Translation: $P \quad = @$

Lorentz transformations: $M \quad\quad = p \; @ \quad\quad p \; @$

Dilation: $D \quad = \quad i \, (p \; @ \quad + \quad\quad)$

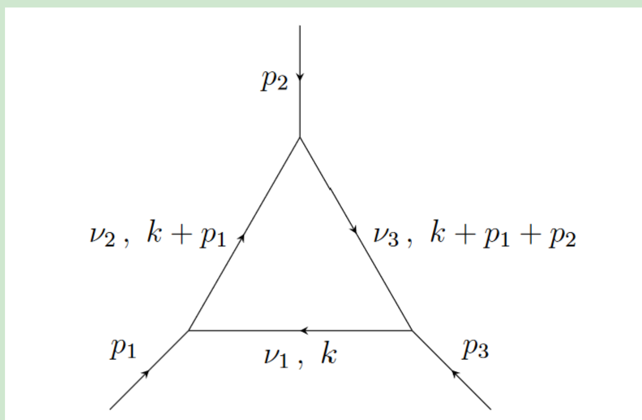
Special conformal boosts: $K \quad = i \;\; p^2 @ \quad\quad 2p \;\; p \; @ \quad\quad 2 \quad p$

Example (Triangle Feynman integral)

Conformally invariant functions of three particles are annihilated by

$$
\begin{aligned}
P_1 &= 4(x_1\partial_1^2 - x_3\partial_3^2) + 4(\partial_1 - \partial_3),\\
P_2 &= 4(x_2\partial_2^2 - x_3\partial_3^2) + 4(\partial_2 - \partial_3),\\
P_3 &= x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + 1.
\end{aligned}
$$

with the change of coordinates $x_1 = p_1^2$, $x_2 = p_2^2$, and $x_3 = p_1 \cdot p_2$.



Main result

Theorem (Henn-P.-Sattelberger-Zoia)

The series solutions to the triangle Feynman integral are

$$
\begin{aligned}
\tilde{f}_1(y_2; y_3) &= 1 + y_2 + y_3 + y_2^2 + 4y_2y_3 + y_3^2 + y_2^3 + 9y_2^2y_3 + y_2^4 + \quad ; \\
\tilde{f}_2(y_2; y_3) &= \log(y_2) + \log(y_2)y_2 + (2 + \log(y_2))y_3 + \log(y_2)y_2^2 + (4 + 4\log(y_2))y_2y_3 \\
&\quad + (3 + \log(y_2))y_3^2 + (\log(y_2))y_2^3 + (6 + 9\log(y_2))y_2^2y_3 + \quad ; \\
\tilde{f}_3(y_2; y_3) &= \log(y_3) + (2 + \log(y_3))y_2 + \log(y_3)y_3 + (3 + \log(y_3))y_2^2 \\
&\quad + (4 + 4\log(y_3))y_2y_3 + \log(y_3)y_3^2 + \quad \frac{11}{3} + \log(y_3) \quad y_2^3 \\
&\quad + (15 + 9\log(y_3))y_2^2y_3 + \quad \frac{25}{6} + \log(y_3) \quad y_2^4 + \quad ; \\
\tilde{f}_4(y_2; y_3) &= \log(y_2)\log(y_3) + (\quad 2 + 2\log(y_2) + \log(y_2)\log(y_3))y_2 \\
&\quad + (\quad 2 + 2\log(y_3) + \log(y_2)\log(y_3))\, y_3 \\
&\quad + \quad \frac{5}{2} + 3\log(y_2) + \log(y_2)\log(y_3) \quad y_2^2 \\
&\quad + (\quad 6 + 4\log(y_2) + 4\log(y_3) + 4\log(y_2)\log(y_3))\, y_2y_3 + \quad :
\end{aligned}
$$

where $y_2 = \frac{x_1}{x_2};\ y_3 = \frac{x_1}{x_3}:$

Q: How to we understand systems of linear PDEs algebraically?

Definition (D-module)

The *nth Weyl algebra*, denoted D_n or D, is the $\mathbb{C}$-algebra

$$D := \mathbb{C}[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle,$$

where all generators commute except ∂_i and x_i, which satisfy the "Leibniz rule"

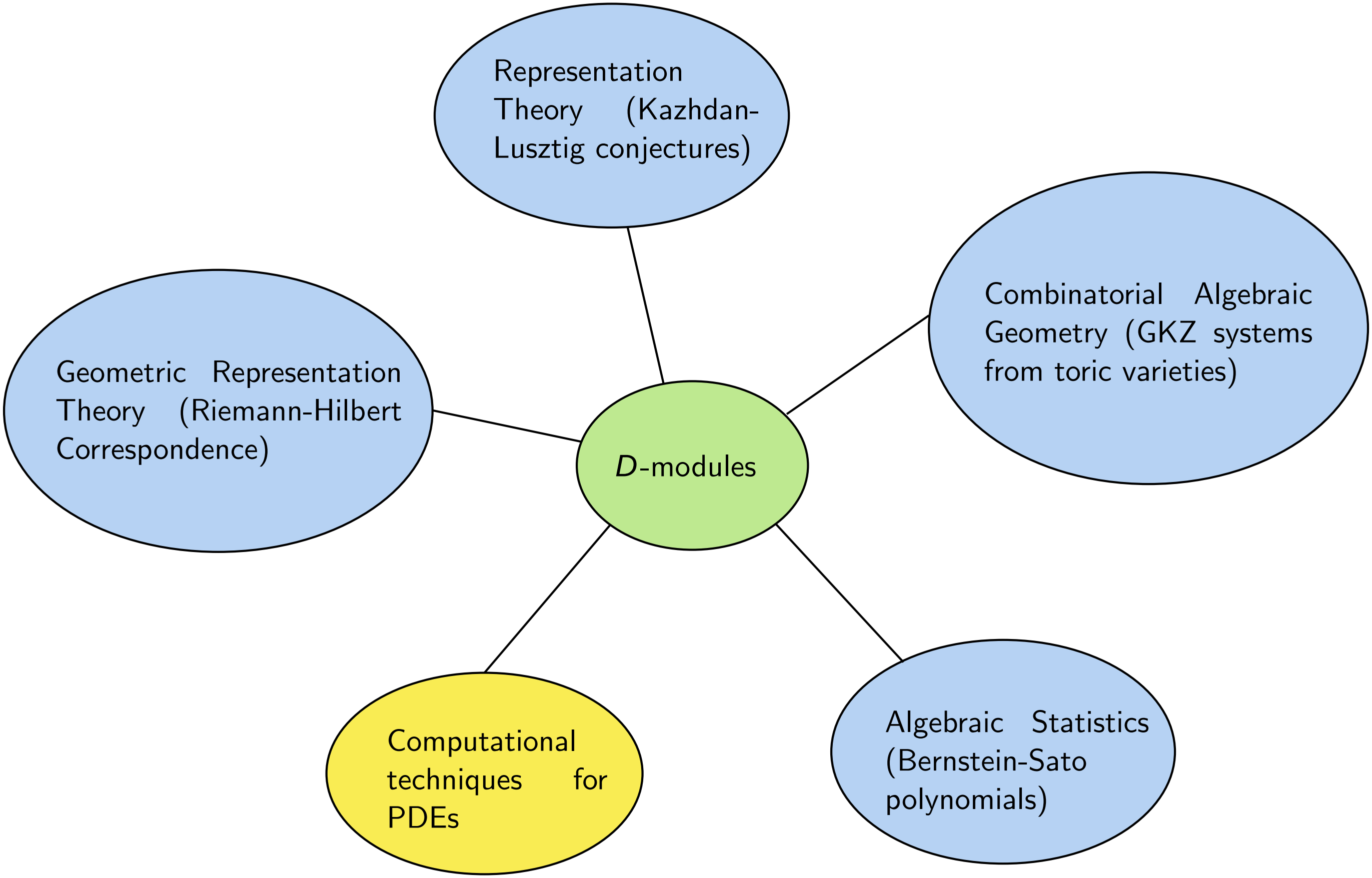
$$[\partial_i, x_i] = \partial_i x_i - x_i \partial_i = 1.$$

A D-module is a module over the Weyl algebra.

Example

Let I be any left D-ideal. Then D/I is D-module.

Why Are D-modules Cool?



Why Are D-modules Useful for PDEs?

Suppose your *D*-ideal is *holonomic*. Then one can compute:

The number of linearly independent holomorphic solutions in a simply connected neighborhood of a generic point (the *holonomic rank*)

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- Where your solutions can have singularities (the *singular locus*)

Why Are D-modules Useful for PDEs?

Suppose your D-ideal is *holonomic*. Then one can compute:

- The number of linearly independent holomorphic solutions in a simply connected neighborhood of a generic point (the *holonomic rank*)
- Where your solutions can have singularities (the *singular locus*)
- Solutions in the form of series expansions

An Example in Macaulay 2

```
i3 : R = QQ[x1, x2, x3]; D = makeWA R;

i5 : q_1 = 4*(x1*dx1^2-x3*dx3^2) + 4*(dx1 - dx3);

i6 : q_2 = 4*(x2*dx2^2-x3*dx3^2) + 4*(dx2 - dx3);

i7 : q_3 = x1*dx1 + x2*dx2 + x3*dx3 + 1;

i8 : I = ideal{q_1, q_2, q_3}
```

```
i9 : isHolonomic(I)

o9 = true

i10 : holonomicRank(I)

o10 = 4

i11 : singLocus(I)

                3            2   2            3         2     2
o11 = ideal(x1 x2*x3 - 2x1 x2 x3 + x1*x2 x3 - 2x1 x2*x3  -
      -----------------------------------------------------
            2   2            3
      2x1*x2 x3  + x1*x2*x3 )
```

Here Sing(I) is the union of a cone and the hyperplanes $x_i = 0$:

$$\text{Sing}(I) = f x_1 x_2 x_3 (x_1^2 + x_2^2 + x_3^2 \quad 2x_1x_2 \quad 2x_1x_3 \quad 2x_2x_3) = 0 g:$$

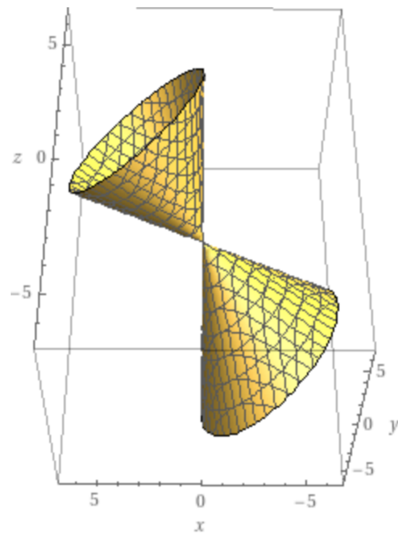


Figure 1: Hypersurface $x_1^2 + x_2^2 + x_3^2 \quad 2x_1x_2 \quad 2x_1x_3 \quad 2x_2x_3$

Solving differential equations?

Q: Given an ODE, how to obtain series solutions?

Example (Frobenius algorithm)

Let $P = \partial^2 + 1$ and guess the solution is of the form $f(x) = \sum_n a_n x^n$. Then we get

$$\begin{aligned} 0 &= P \circ f(x) \\ &= \sum_n n(n-1)a_n x^{n-2} - \sum_n a_n x^n \\ &= \sum_n (n+2)(n+1)a_{n+2}x^n - \sum_n a_n x^n \\ &= \sum_n ((n+2)(n+1)a_{n+2} - a_n)x^n. \end{aligned}$$

So $a_{n+2} = \frac{1}{(n+1)(n+2)}a_n$, and plugging in initial values we recover the power series for $\sin(x), \cos(x)$.

The SST (Saito, Sturmfels, Takayama) algorithm

Inputs:

- A regular holonomic D-ideal I
- A direction w to expand in, which is in the interior of a Gröbner cone C_w of I

Output:

- A list of starting monomials
- For each starting monomial $x^A \log(x)^B$, with A 2 C and B 2 Z; a *Nilsson series*

$$x^A \sum_{p;\ 0 \quad b_i \quad \mathrm{rk}(I) 2 Z} c_{pb} x^p \log(x)^b$$

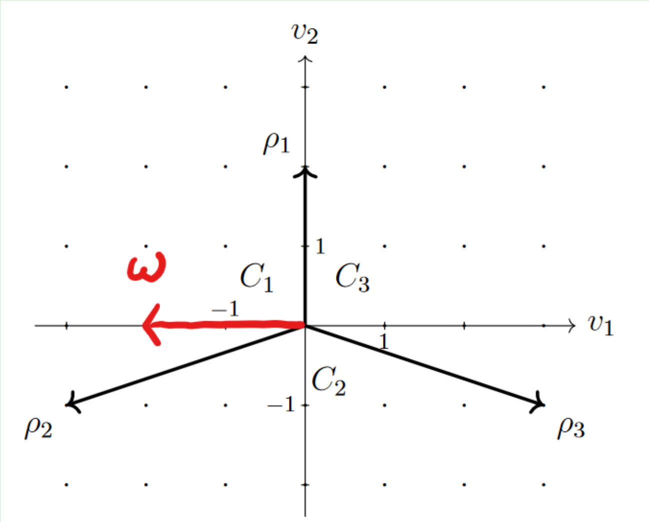
which converges for z such that $(\quad \log(jz_1j); :::; \quad \log(jz_nj))$ is in a translate of C_w:

Source: Grobner Deformations of Hypergeometric Di erential Equations (Mutsumi Saito, Bernd Sturmfels, Nobuki Takayama)

Example of SST

Example (Scattering function of three particles)

The Gröbner fan lives in $\mathbb{R}^3 = \mathbb{R}(1;1;1)$ and looks like



The starting monomials are:

$$x_1^{-1};\ x_1^{-1}\log\left(\frac{x_1}{x_2}\right);\ x_1^{-1}\log\left(\frac{x_1}{x_3}\right);\ x_1^{-1}\log\left(\frac{x_1}{x_2}\right)\log\left(\frac{x_1}{x_3}\right):$$

Main result

Theorem (Henn-P.-Sattelberger-Zoia)

The series solutions to the triangle Feynman integral are

$$\tilde{f}_1(y_2; y_3) = 1 + y_2 + y_3 + y_2^2 + 4y_2y_3 + y_3^2 + y_2^3 + 9y_2^2y_3 + y_2^4 + \quad ;$$

$$\begin{aligned}\tilde{f}_2(y_2; y_3) = {} & \log(y_2) + \log(y_2)y_2 + (2 + \log(y_2))y_3 + \log(y_2)y_2^2 + (4 + 4\log(y_2))y_2y_3 \\ & + (3 + \log(y_2))y_3^2 + (\log(y_2))y_2^3 + (6 + 9\log(y_2))y_2^2y_3 + \quad ;\end{aligned}$$

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where $y_2 = \frac{x_1}{x_2}; y_3 = \frac{x_1}{x_3}:$

Solutions from physics:

$$f_1(x_1; x_2; x_3) = \frac{1}{\sqrt{}}\Big[\mathrm{Li}_2(\ _2) + \mathrm{Li}_2(\ _3) + \frac{\ ^2}{6} + \frac{1}{2}\log\left(\frac{\ _3}{\ _2}\right)\log\left(\frac{1\ \ _3}{1\ \ _2}\right) + \frac{1}{2}\log(\ \ _2)\log(\ \ _3)\Big];$$

$$f_2(x_1; x_2; x_3) = \frac{1}{\sqrt{}}\log\left(\frac{x_1\ \ x_2\ \ x_3\ \ \sqrt{}}{x_1\ \ x_2\ \ x_3 + \sqrt{}}\right);$$

$$f_3(x_1; x_2; x_3) = \frac{1}{\sqrt{}}\log\left(\frac{x_2\ \ x_1\ \ x_3\ \ \sqrt{}}{x_2\ \ x_1\ \ x_3 + \sqrt{}}\right);$$

$$f_4(x_1; x_2; x_3) = \frac{1}{\sqrt{}};$$

where

$$\ _2 = \frac{2x_2}{(x_1\ \ x_2\ \ x_3\ \ \sqrt{})}; \qquad \ _3 = \frac{2x_3}{(x_1\ \ x_2\ \ x_3\ \ \sqrt{})}:$$

Outlook

Can D-module techniques be used to find scattering amplitudes of systems with more particles? (Involves finding more differential equations)

Can we use SST to catalogue series expansions of multivariate functions?