The Segre Determinant

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Slides: https://lizziepratt.com/notes
arXiv 2505.09204

June 12, 2025

When do six points in $\mathbb{P}^2$ lie on a conic?

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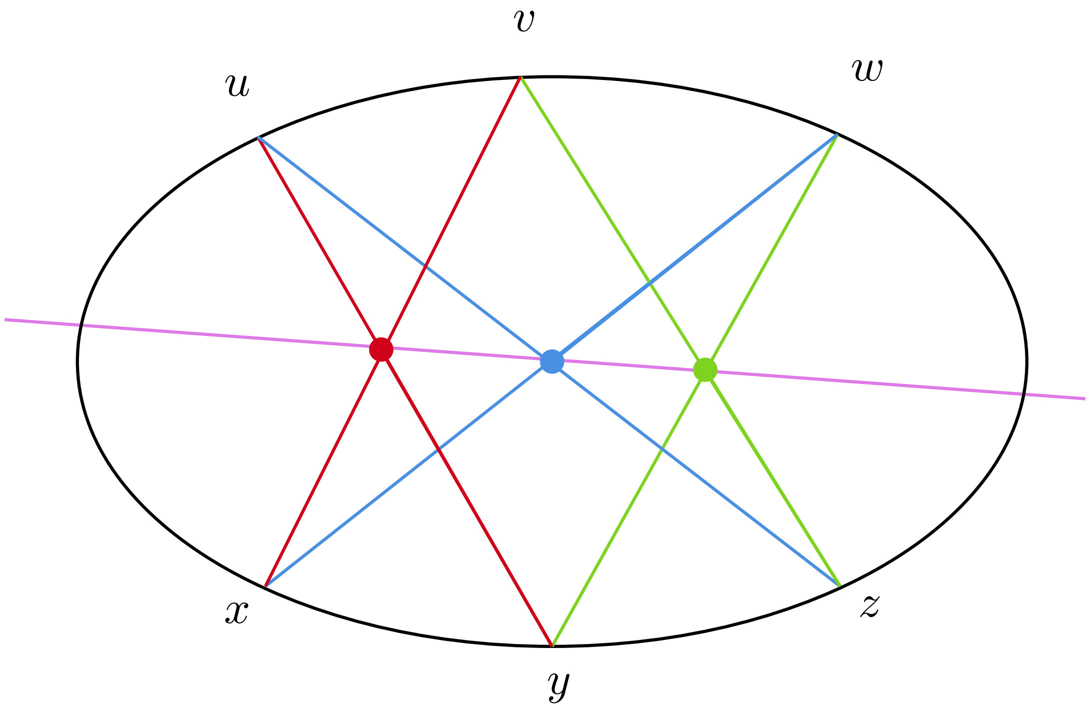
1. Pascal's theorem (1640) and its converse (Coxeter-Greitzer 1967):



2. Determinant of the following vanishes:

$$\begin{bmatrix} u_0^2 & v_0^2 & w_0^2 & x_0^2 & y_0^2 & z_0^2 \\ u_0u_1 & v_0v_1 & w_0w_1 & x_0x_1 & y_0y_1 & z_0z_1 \\ u_0u_2 & v_0v_2 & w_0w_2 & x_0x_2 & y_0y_2 & z_0z_2 \\ u_1^2 & v_1^2 & w_1^2 & x_1^2 & y_1^2 & z_1^2 \\ u_1u_2 & v_1v_2 & w_1w_2 & x_1x_2 & y_1y_2 & z_1z_2 \\ u_2^2 & v_2^2 & w_2^2 & x_2^2 & y_2^2 & z_2^2 \end{bmatrix}$$

i.e. the images under the Veronese map $v_2 : \mathbb{P}^2 \to \mathbb{P}^5$ lie on a hyperplane.

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$$q_{012}\, q_{034}\, q_{135}\, q_{245} - q_{013}\, q_{024}\, q_{125}\, q_{345} = 0.$$

When do points lie on a hypersurface?

When do $\binom{n+d-1}{d}$ in $\mathbb{P}^{n-1}$ lie on a hypersurface of degree d?

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Today: when do $k\ell$ points lie on a bilinear hypersurface in $\mathbb{P}^{k-1} \times \mathbb{P}^{l-1}$?

The Segre determinant

Fix $n = kl$ and let $A_1 \times B_1, \ldots, A_n \times B_n$ denote n points in $\mathbb{P}^{k-1} \times \mathbb{P}^{l-1}$. The *Segre determinant* is the polynomial

$$\mathsf{Seg}_{k,\ell} = \det \begin{bmatrix} \vdots & & \vdots \\ A_1 \otimes B_1 & \ldots & A_n \otimes B_n \\ \vdots & & \vdots \end{bmatrix}.$$

Example

Four points $\begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} \times \begin{bmatrix} b_{1,1} \\ b_{2,1} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1,4} \\ a_{2,4} \end{bmatrix} \times \begin{bmatrix} b_{1,4} \\ b_{2,4} \end{bmatrix} \in \mathbb{P}^1 \times \mathbb{P}^1$. Then

$$\mathsf{Seg}_{2,2} = \det \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,2}b_{1,2} & a_{1,3}b_{1,3} & a_{1,4}b_{1,4} \\ a_{1,1}b_{2,1} & a_{1,2}b_{2,2} & a_{1,3}b_{2,3} & a_{1,4}b_{2,4} \\ a_{2,1}b_{1,1} & a_{2,2}b_{1,2} & a_{2,3}b_{1,3} & a_{2,4}b_{1,4} \\ a_{2,1}b_{2,1} & a_{2,2}b_{2,2} & a_{2,3}b_{2,3} & a_{2,4}b_{2,4} \end{bmatrix}.$$

Example, continued

Lemma

The Segre determinant $\mathrm{Seg}_{k,\ell}$ *is a polynomial of bi-degree* (l,k) *in the maximal minors of*

$$A \ := \begin{bmatrix} A_1 & \dots & A_n \end{bmatrix}, \qquad B \ := \begin{bmatrix} B_1 & \dots & B_n \end{bmatrix}.$$

Example

In maximal minors of

$$A \ := \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \end{bmatrix}, \qquad B \ := \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \end{bmatrix}$$

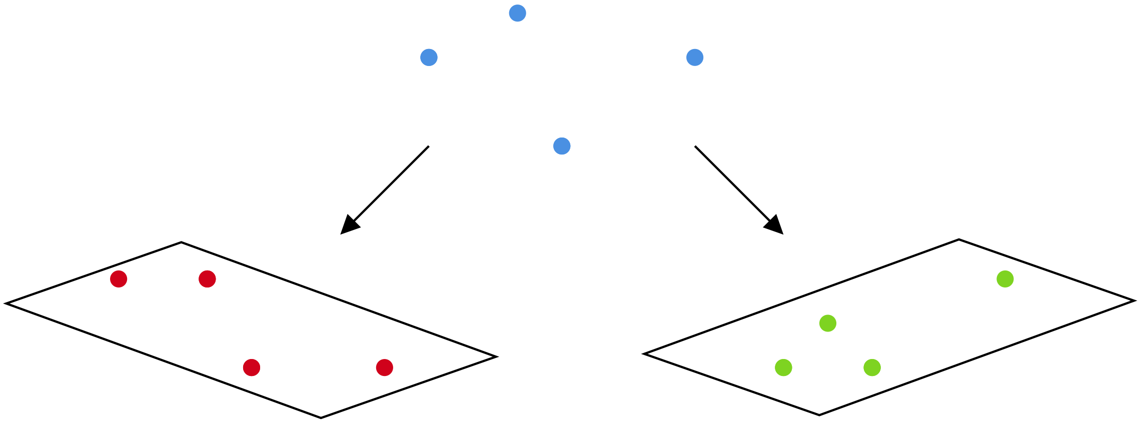
we have

$$\mathrm{Seg}_{2,2} = A_{12}A_{34}B_{13}B_{24} - A_{13}A_{24}B_{12}B_{34}.$$

Vanishes when the *cross-ratios* $\frac{A_{12}A_{34}}{A_{13}A_{24}}$ and $\frac{B_{12}B_{34}}{B_{13}B_{24}}$ are equal.

Computer vision

The polynomial $\text{Seg}_{3,3}$ appears in *algebraic vision* as a necessary condition for two configurations of nine points in $\mathbb{P}^2$ to have a common recovery in $\mathbb{P}^3$.



See [*On The Existence of Epipolar Matrices*, ALST 2016]

$$
\begin{aligned}
\mathrm{Seg}_{3,3} = {} & [123][456][789](3\langle 123\rangle\langle 457\rangle\langle 689\rangle-\langle 123\rangle\langle 467\rangle\langle 589\rangle+3\langle 124\rangle\langle 356\rangle\langle 789\rangle-3\langle 124\rangle\langle 357\rangle\langle 689\rangle+\langle 124\rangle\langle 367\rangle\langle 589\rangle- \\
& \langle 124\rangle\langle 368\rangle\langle 579\rangle-\langle 125\rangle\langle 346\rangle\langle 789\rangle+\langle 125\rangle\langle 347\rangle\langle 689\rangle+\langle 127\rangle\langle 348\rangle\langle 569\rangle-\langle 134\rangle\langle 258\rangle\langle 679\rangle-\langle 135\rangle\langle 247\rangle\langle 689\rangle+ \\
& \langle 145\rangle\langle 267\rangle\langle 389\rangle+\langle 147\rangle\langle 258\rangle\langle 369\rangle)+[123][457][689](-3\langle 123\rangle\langle 456\rangle\langle 789\rangle+\langle 124\rangle\langle 368\rangle\langle 579\rangle-\langle 126\rangle\langle 348\rangle\langle 579\rangle+ \\
& \langle 135\rangle\langle 246\rangle\langle 789\rangle-\langle 146\rangle\langle 258\rangle\langle 379\rangle)+[123][458][679](-\langle 124\rangle\langle 367\rangle\langle 589\rangle-\langle 125\rangle\langle 346\rangle\langle 789\rangle+\langle 126\rangle\langle 347\rangle\langle 589\rangle+ \\
& \langle 146\rangle\langle 257\rangle\langle 389\rangle)+[123][467][589](\langle 123\rangle\langle 456\rangle\langle 789\rangle-\langle 124\rangle\langle 358\rangle\langle 679\rangle+\langle 125\rangle\langle 348\rangle\langle 679\rangle+\langle 134\rangle\langle 256\rangle\langle 789\rangle- \\
& \langle 135\rangle\langle 246\rangle\langle 789\rangle+\langle 145\rangle\langle 268\rangle\langle 379\rangle)+[124][356][789](-3\langle 123\rangle\langle 456\rangle\langle 789\rangle+\langle 123\rangle\langle 468\rangle\langle 579\rangle+\langle 135\rangle\langle 247\rangle\langle 689\rangle- \\
& \langle 135\rangle\langle 267\rangle\langle 489\rangle-\langle 137\rangle\langle 258\rangle\langle 469\rangle)+[124][357][689](3\langle 123\rangle\langle 456\rangle\langle 789\rangle-\langle 123\rangle\langle 468\rangle\langle 579\rangle-\langle 135\rangle\langle 246\rangle\langle 789\rangle+ \\
& \langle 136\rangle\langle 258\rangle\langle 479\rangle)+[124][358][679](\langle 123\rangle\langle 467\rangle\langle 589\rangle-\langle 136\rangle\langle 257\rangle\langle 489\rangle)+[124][367][589](-\langle 123\rangle\langle 456\rangle\langle 789\rangle+ \\
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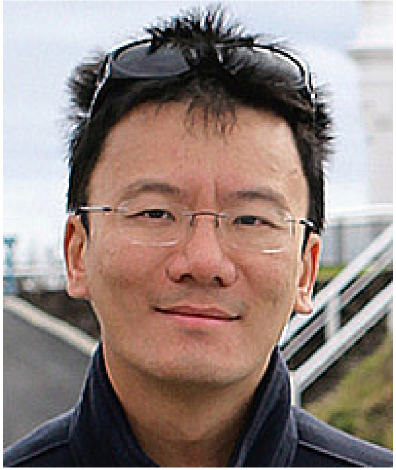
\end{aligned}
$$

The Chow-Lam form (P-Sturmfels 2025)



- The Chow form (1937): Assigns to $\mathcal{V} \subset \mathbb{P}^{n-1}$ a polynomial $C_{\mathcal{V}}$ which encodes it
- Chow-Lam form (2025): Assigns to $\mathcal{V} \subset \mathrm{Gr}(k, n)$ a polynomial $CL_{\mathcal{V}}$ which (usually) encodes it

Thomas Lam studied $CL_{\mathcal{V}}$ where $\mathcal{V}$ is a positroid variety.

The Chow-Lam form, cont.

Fix a variety $\mathcal{V} \subset \mathrm{Gr}(k,n)$ of dimension $k(r-k)-1$. We have

$$\begin{array}{ccc} & \mathrm{Fl}(k, n-r+k, n) & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \mathrm{Gr}(k,n) & & \mathrm{Gr}(n-r+k, n) \end{array}$$

The *Chow-Lam form $CL_{\mathcal{V}}$* cuts out the hypersurface* $\pi_2(\pi_1^{-1}(\mathcal{V}))$.

Dimension-checking:

- Fiber of π_1 is $\mathrm{Gr}(n-r, n)$
- $\dim \pi_1^{-1}(\mathcal{V}) = (r-k)(n-r+k)-1 = \dim \mathrm{Gr}(n-r+k, n) - 1$

*Usually a hypersurface, depends on coefficient of a certain Schubert class in the cohomology expansion

The twisted cubic

We have $k = 1$ and $\dim \mathcal{V} = 1$, so

$$\begin{array}{ccc} & \mathsf{Fl}(1,2,4) & \\ & \swarrow^{\pi_1} \quad \searrow^{\pi_2} & \\ \mathcal{V} \subset \mathbb{P}^3 = \mathsf{Gr}(1,4) & & \mathsf{Gr}(2,4) \end{array}$$

The Chow locus is lines in $\mathbb{P}^3$ which contain a point on $\mathcal{V}$. The Chow form is the determinant of the *Bézout matrix* :

$$C_{\mathcal{V}} = \det \begin{bmatrix} p_{12} & p_{13} & p_{14} \\ p_{13} & p_{14} + p_{23} & p_{24} \\ p_{14} & p_{24} & p_{34} \end{bmatrix}.$$

Its expansion is

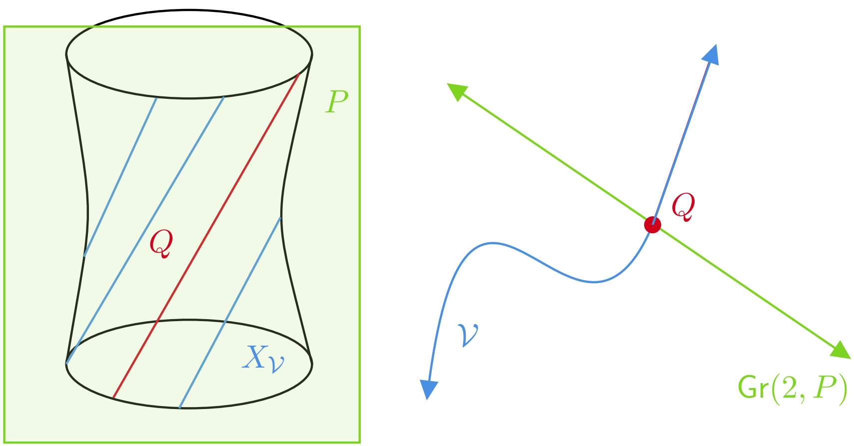
$$-p_{14}^3 - p_{14}^2 p_{23} + 2p_{13}p_{14}p_{24} - p_{12}p_{24}^2 - p_{13}^2 p_{34} + p_{12}p_{14}p_{34} + p_{12}p_{23}p_{34}.$$

Curve in $\mathrm{Gr}(2,4)$

Let $\mathcal{V}$ be a curve in $\mathrm{Gr}(2,4)$, so

$$\begin{array}{ccc} & \mathrm{Fl}(2,3,4) & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \mathcal{V} \subset \mathrm{Gr}(2,4) & & \mathrm{Gr}(3,4) = (\mathbb{P}^3)^\vee \end{array}$$

Then $\mathrm{CL}_{\mathcal{V}}$ is planes P containing a line L in $\mathbb{P}^3$, with L on $\mathcal{V}$.



Let $X_{\mathcal{V}}$ be the surface in $\mathbb{P}^3$ swept out by all of the lines in $\mathcal{V}$.
Then $\mathcal{CL}_{\mathcal{V}}$ equals the dual variety $X_{\mathcal{V}}^\vee$.

Coordinate Systems

A linear space L can represented multiple ways.

- Primal : as the kernel of an $(n-k) \times n$ matrix
- Dual : as the rowspan of a $k \times n$ matrix

The primal and dual Plücker coordinates are the maximal minors of these matrices.

Example (Coordinates on Gr$(3,5)$)

p_{12}	p_{13}	p_{14}	p_{15}	p_{23}	p_{24}	p_{25}	p_{34}	p_{35}	p_{45}
q_{345}	$-q_{245}$	q_{235}	$-q_{234}$	q_{145}	$-q_{135}$	q_{134}	q_{125}	$-q_{124}$	q_{123}

Chow-Lam for torus orbits

The torus $T = (\mathbb{C}^*)^n$ acts on $\mathsf{Gr}(k,n)$ via

$$t \cdot \begin{bmatrix} \vdots & & \vdots \\ A_1 & \dots & A_n \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & & \vdots \\ t_1 A_1 & \dots & t_n A_n \\ \vdots & & \vdots \end{bmatrix}.$$

We denote the orbit closure of a point A in $\mathsf{Gr}(k,n)$ by

$$\mathcal{T}_A := \overline{T \cdot A} \subset \mathsf{Gr}(k,n).$$

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Theorem (P- 2025)

Suppose $n = k\ell$ with $k, \ell \geq 2$ and that $A \in \mathrm{Gr}(k, n)$ has nonzero Plücker coordinates. Then

$$\mathcal{CL}_{\mathcal{T}_A} \subset \mathrm{Gr}(n - \ell, n).$$

The Chow-Lam form of $\mathcal{T}_A$ in primal Plücker coordinates B_I on $\mathrm{Gr}(n - \ell, n)$ is the Segre determinant $\mathrm{Seg}_{k,\ell}(A, B)$.

Example: torus orbits

Fix a general point

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \end{bmatrix} \in \ \mathsf{Gr}(2,6).$$

Then $\mathcal{T}_A$ is a toric variety with polytope

$$\Delta(2,6) = \mathsf{conv}\{110000, 101000, \ldots\} \subset \mathbb{Z}^6.$$

Its Chow-Lam form is

$$\mathsf{Seg}_{2,3} = (A_{12}A_{34}A_{56} + A_{14}A_{25}A_{36})\, B_{123}B_{456} - A_{13}A_{25}A_{46}\, B_{124}B_{356}$$
$$+A_{12}A_{35}A_{46}\, B_{134}B_{256} - A_{12}A_{34}A_{56}\, B_{135}B_{246} + A_{13}A_{24}A_{56}\, B_{125}B_{346}.$$

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- Cohomology class?

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- Dimension of X? Codimension in $\mathrm{Gr}(2,6)$?
- Cohomology class? $4\Omega_3 + 2\Omega_{2,1}$
- Degree in $\mathbb{P}^{14}$?

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$$\Delta(2,6) = \mathrm{conv}\{110000, 101000, \ldots\} \subset \mathbb{Z}^6.$$

Its Chow-Lam form is

$$\mathrm{Seg}_{2,3} = (A_{12}A_{34}A_{56} + A_{14}A_{25}A_{36})\, B_{123}B_{456} - A_{13}A_{25}A_{46}\, B_{124}B_{356}$$
$$+A_{12}A_{35}A_{46}\, B_{134}B_{256} - A_{12}A_{34}A_{56}\, B_{135}B_{246} + A_{13}A_{24}A_{56}\, B_{125}B_{346}.$$

- Dimension of X? Codimension in $\mathrm{Gr}(2,6)$?
- Cohomology class? $4\Omega_3 + 2\Omega_{2,1}$
- Degree in $\mathbb{P}^{14}$? 26

Degree formula

Suppose $\dim \mathcal{V} = k(r-k) - 1$ and write

$$[\mathcal{V}] = \sum_{\lambda \subset k \times (n-k)} c_\lambda(\mathcal{V}) \cdot [\Omega_\lambda] \ \in \ H^*(\mathsf{Gr}(2,4), \mathbb{Z})$$

where

$$\Omega_\lambda = \{L \ : \ L \cap E_i \geq n - k + \lambda_i - i\}.$$

The *Chow-Lam degree* $\alpha(\mathcal{V})$ is the unique coefficient with

$$\lambda = (n - r + 1, n - r, \ldots, n - r).$$

Proposition (P-Sturmfels 2025)

The Chow-Lam form $CL_{\mathcal{V}}$ is an irreducible polynomial of degree $\alpha(\mathcal{V})$.

Non-generic torus orbits

Theorem (P- 2025)

Fix a point A in the Grassmannian Gr(k, n) such that $\dim \mathcal{T}_A = k(r-k) - 1 < n$. Then the Chow-Lam form of $\mathcal{T}_A$ in primal Plücker coordinates B_I on Gr$(n-r+k, n)$ divides the Segre determinant Seg$_{k,\ell}(A, B)$

Example

Suppose that A is in Gr$(2, 4)$ and $A_{12} = 0$. Then the 4×4 Segre matrix has determinant $A_{13}A_{34}B_{12}B_{34}$, but the Chow-Lam form is B_{12}.

Klyachko's formula

The variety $\mathcal{T}_A$ depends only on the *matroid* M of A, i.e.

$$\{I \ : \ A_I \neq 0\} \subset \binom{[n]}{k}.$$

The numbers $c_\lambda(M) := c_\lambda(A)$ are the *Schubert coefficients of M.*

Proposition (Klyachko 85)

Let λ be a partition fitting in a $k \times (n-k)$ rectangle. Then the coefficient $c_\lambda(U_{k,n})$ is

$$c_\lambda(U_{k,n}) \ = \ \sum_{i=0}^{k} (-1)^i \binom{n}{i} \dim \mathbb{S}_{\lambda^c}(\mathbb{C}^{k-i}). \qquad (1)$$

Corollary

The Chow-Lam degree $\alpha(U_{k,n})$ is k.

Chow varieties

Setup and notation:

- A nonsingular projective variety X
- The set $\mathcal{C}_r(X, \delta)$ of dimension r cycles with class δ in singular homology

Choose an embedding $\iota : X \hookrightarrow \mathbb{P}(V)$ and let $d := \iota_* \delta$ in $H_{2r}(\mathbb{P}(V), \mathbb{Z}) \cong \mathbb{Z}$. Then

$$\begin{aligned} \mathcal{C}_r(X, \delta) \;\subset\; \mathcal{C}_r(\mathbb{P}(V), d) \;&\xrightarrow{\varphi} \mathbb{P}(\mathsf{Sym}^d(\wedge^{\dim V - r - 1} V)) \\ \mathcal{V} \;&\mapsto C_{\mathcal{V}}. \end{aligned}$$

Definition

We call $\overline{\varphi(\mathcal{C}_r(X, \delta))}$ the *Chow variety of r-cycles with class δ*.

Independent of embedding ι by [Barlet 1975] .

Chow quotients

Setup:

- A nonsingular projective variety X
- An algebraic group H acting on X
- A H-stable subset $U \subset X$ where the dimension and cohomology class of $\overline{H \cdot x}$ are constant

Idea: create parameter space for H-orbits which is a projective variety.

Definition

The *Chow quotient* $X//H$ is the closure of the image of

$$\begin{aligned} U &\to \mathcal{C}_r(X, \delta) \\ x &\mapsto \overline{H \cdot x}. \end{aligned}$$

Example of Chow quotient

Setup:

- $(\mathbb{C}^*)^6$ acting on $\mathrm{Gr}(2,6)$
- U is the collection of points x with nonzero Plücker coordinates
- $\iota : \mathrm{Gr}(2,6) \hookrightarrow \mathbb{P}^{14}$ is the Plücker embedding
- $\delta = 4\Omega_3 + 2\Omega_2$ and $d = 26$

Then

$$\mathrm{Gr}(2,6)//(\mathbb{C}^*)^6 \hookrightarrow \mathbb{P}(\mathrm{Sym}^{26}(\wedge^8\mathbb{C}^{14})).$$

The right side is very big!! Dimension roughly 10^{63}.

The Segre coefficient variety

Instead: encode torus orbits by their Chow-Lam form. Recall

$$\mathrm{Seg}_{2,3} = (A_{12}A_{34}A_{56} + A_{14}A_{25}A_{36})\, B_{123}B_{456} - A_{13}A_{25}A_{46}\, B_{124}B_{356}$$
$$+A_{12}A_{35}A_{46}\, B_{134}B_{256} - A_{12}A_{34}A_{56}\, B_{135}B_{246} + A_{13}A_{24}A_{56}\, B_{125}B_{346}.$$

So we consider

$$\mathrm{Gr}(2,6)^\circ \to \mathbb{P}^4$$
$$A \mapsto (A_{12}A_{34}A_{56} + A_{14}A_{25}A_{36},\ -A_{13}A_{25}A_{46},$$
$$A_{12}A_{35}A_{46},\ -A_{12}A_{34}A_{56},\ A_{13}A_{24}A_{56}).$$

The image is the Segre threefold cut out by

$$x_0x_1x_3 - x_1x_2x_3 - x_0x_2x_4 - x_1x_2x_4 - x_1x_3x_4 - x_2x_3x_4.$$

It is the unique (up to isomorphism) cubic hypersurface in $\mathbb{P}^4$ with the maximum number of ordinary double points, namely ten.

The Serge coefficient variety

Consider the map

$$\pi : \mathsf{Gr}(k, k\ell)^\circ \to \mathbb{P}(H^0(\mathcal{O}_{\mathsf{Gr}(\ell,k\ell)}(k))$$
$$A \mapsto \mathsf{Seg}(A, B).$$

We call the image the *Segre coefficient variety* .

Theorem (P. 2025)

The Segre coefficient variety with coefficients in A is a projective variety of dimension $k(k\ell - k - 1) + 1$ which is birationally equivalent to the Chow quotient $Gr(k, k\ell)//T$.

Thank you for listening!

Bonus: proof idea

Fix $A \in \mathrm{Gr}(2,6)$. Parameterize the CL locus of $\mathcal{T}_A$ as 3×6 matrices B such that, for some $t \in (\mathbb{C}^*)^6$, we have

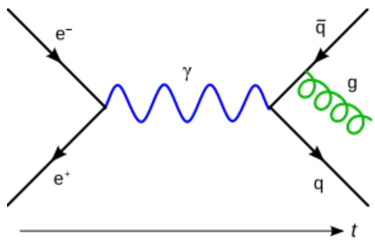
$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \end{bmatrix} \cdot \mathrm{diag}(t_1, ..., t_6) \cdot \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \\ a_{14} & a_{24} \\ a_{15} & a_{25} \\ a_{16} & a_{26} \end{bmatrix} = 0.$$

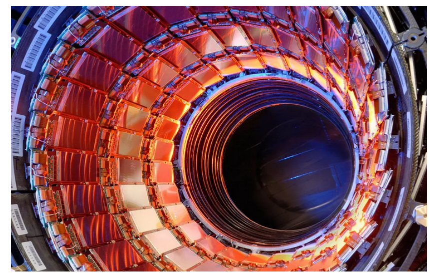
Re-arranging, we obtain the expression

$$t_1 \begin{bmatrix} a_{11}b_{11} \\ a_{11}b_{21} \\ a_{11}b_{31} \\ a_{21}b_{11} \\ a_{21}b_{21} \\ a_{21}b_{31} \end{bmatrix} + t_2 \begin{bmatrix} a_{12}b_{12} \\ a_{12}b_{22} \\ a_{12}b_{32} \\ a_{22}b_{12} \\ a_{22}b_{22} \\ a_{22}b_{32} \end{bmatrix} + \ldots + t_6 \begin{bmatrix} a_{16}b_{16} \\ a_{16}b_{26} \\ a_{16}b_{36} \\ a_{26}b_{16} \\ a_{26}b_{26} \\ a_{26}b_{36} \end{bmatrix} = \sum_{i=1}^{6} t_i A_i \otimes B_i = 0.$$

Positive Geometry

- Math context: "positive" parts of varieties, e.g. $\mathrm{Gr}^{\geq 0}(k,n)$, $Fl^{\geq 0}(n)$, $\mathcal{M}_{0,n}^{\geq 0}$ ...
- Physics context: computing probabilities in particle scattering





Example

The positive Grassmannian $\mathrm{Gr}^{\geq 0}(k,n) := \mathrm{Gr}(k,n) \cap \mathbb{P}_{\mathbb{R}}^{\binom{n}{k}-1}$. Its boundaries are known as *positroid varieties.*

See [*Positive Geometries and Canonical Forms*, AHBL]