Exterior Cyclic Polytopes and Convexity of Amplituhedra

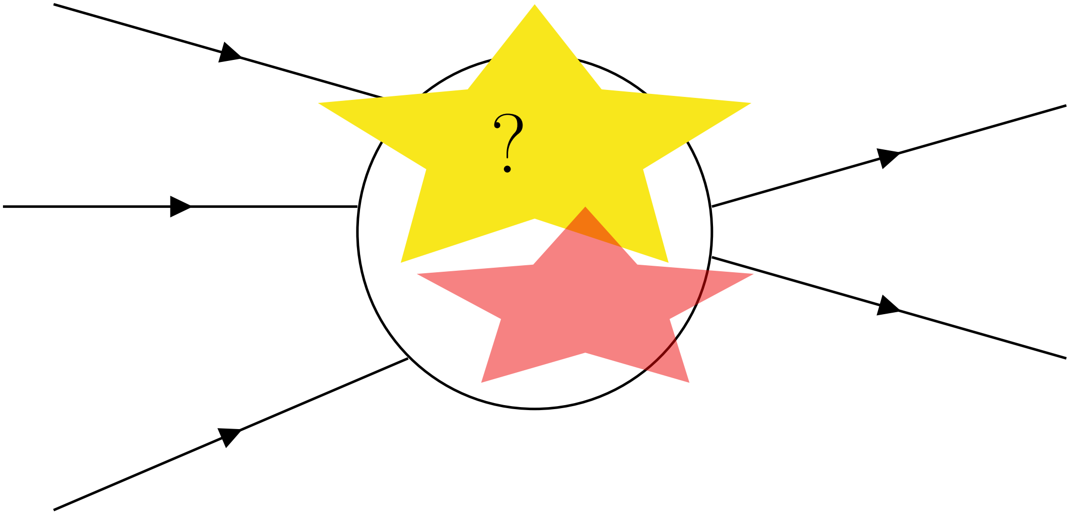
Lizzie Pratt

Joint with Elia Mazzucchelli
https://lizziepratt.com/notes

October 28, 2025

Goal: predict outcome of particle collisions

$\rightsquigarrow$ scattering amplitude.



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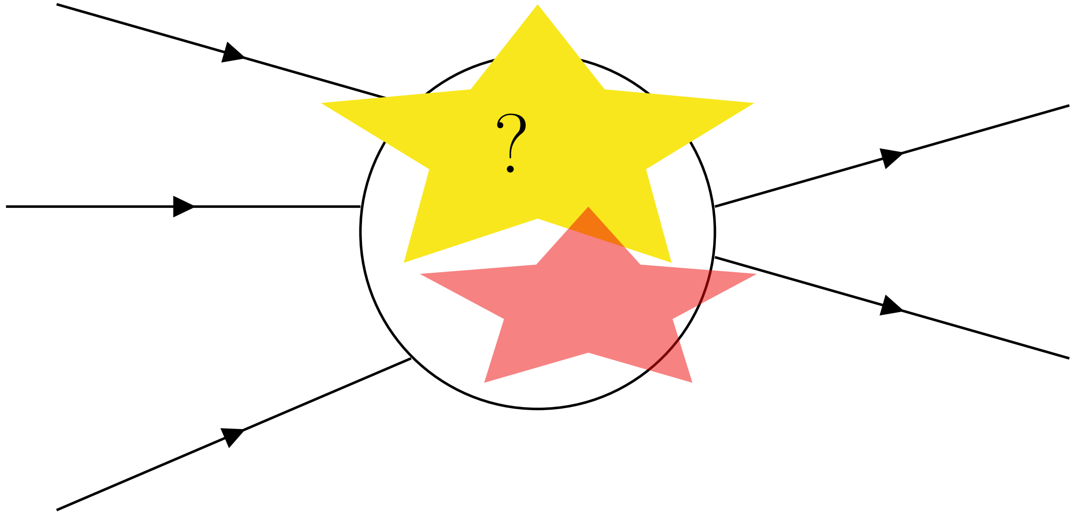

Problem: impossible to compute.

background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary $2 \to 4$ processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is

Parke and Taylor, *An amplitude for n gluon scattering* (1986):

$$|\mathcal{M}_n(--+++\ldots)|^2 = c_n(g,N)\ \big[\ (1\cdot 2)^4 \sum_P \frac{1}{(1\cdot 2)(2\cdot 3)(3\cdot 4)\ldots(n\cdot 1)} + \mathcal{O}(N^{-2}) + \mathcal{O}(g^2)\ \big]$$

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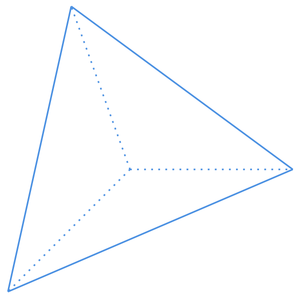
Arkani-Hamed and Trnka, *The Amplituhedron* (2013): amplitudes in $\mathcal{N}=4$ super Yang-Mills are "volumes" of geometric objects!

The *projective simplex* is

$$\Delta_n := \mathrm{conv}\{e_0, \ldots, e_n\} \subset \mathbb{P}^n.$$



The *Grassmannian* parameterizes k-spaces in $\mathbb{R}^n$, and is a projective variety via

$$\begin{aligned} \mathrm{Gr}(k,n) &\to \mathbb{P}(\wedge^k \mathbb{R}^n) \\ \mathrm{span}(v_1, \ldots, v_k) &\mapsto v_1 \wedge \ldots \wedge v_k. \end{aligned}$$

The *positive Grassmannian* is

$$\mathrm{Gr}_{\geq 0}(k,n) := \Delta_{\binom{n}{k}-1} \cap \mathrm{Gr}(k,n).$$

Let Z be a $(k+m) \times n$ matrix with positive maximal minors.

$$\begin{aligned} \wedge^k Z : \; Gr(k,n) \; &\dashrightarrow \; Gr(k, k+m) \\ \mathsf{span}(v_1, \ldots, v_k) \; &\mapsto \mathsf{span}(Zv_1, \ldots, \wedge Z v_k). \end{aligned}$$

The *amplituhedron* $\mathcal{A}_{k,m,n}(Z)$ is the image of $\mathrm{Gr}_{\geq 0}(k,n)$.

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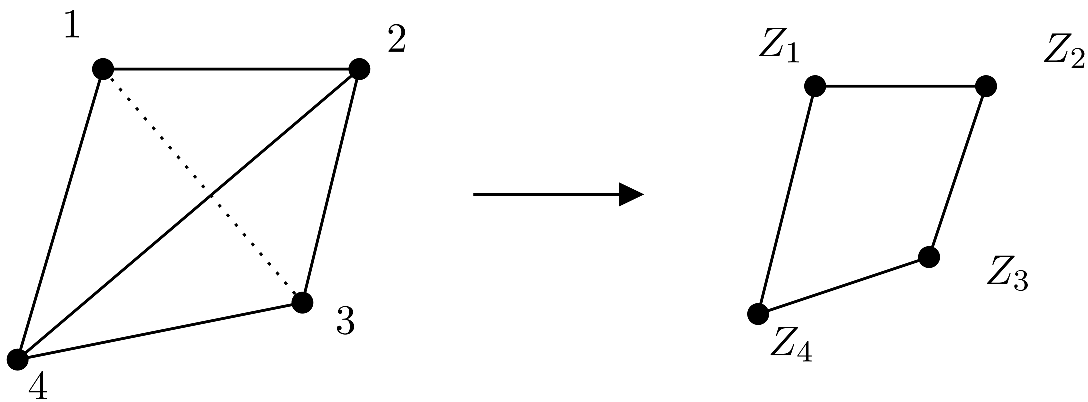
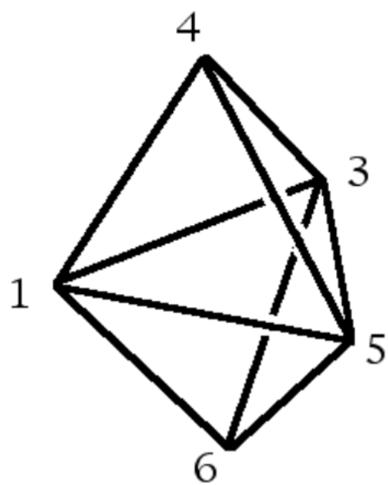
Example $(k=1)$

$$Z : \Delta_{n-1} \to \mathbb{P}^m$$
$$e_i \mapsto Z_i$$



The image is a *cyclic polytope*.

Some cyclic polytopes in $\mathbb{P}^3$:



[Hodges 2009]

Gr$_{\geq 0}(k,n)$: linear (simplex) $\cap$ nonlinear (Grassmannian).
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The *twistor coordinates* wrt Z on $\mathsf{Gr}(k, k+2)$ are

$$\langle ij \rangle := \det[Z_i \, Z_j \, Y^T], \qquad [Y] \in \mathsf{Gr}(k, k+2).$$

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On $\mathsf{Gr}(2, 4)$, we have

$$\langle 12 \rangle = (z_{1i}z_{2j} - z_{2i}z_{1j})p_{34} - (z_{1i}z_{3j} - z_{3i}z_{1j})p_{24} + (z_{2i}z_{3j} - z_{3i}z_{2j})p_{14} + \dots$$

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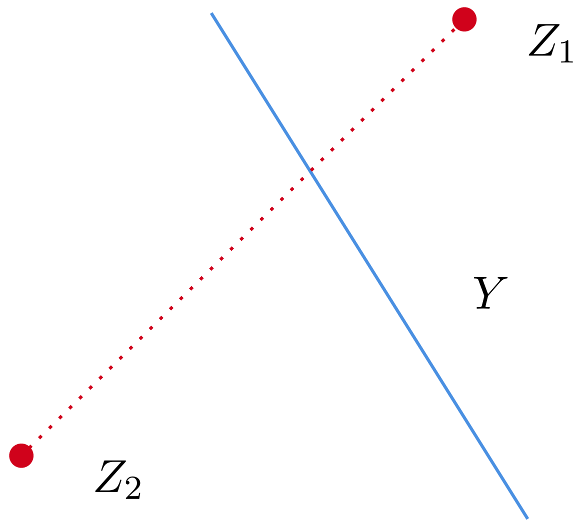
This vanishes on lines $[Y]$ meeting the line $\overline{Z_1 Z_2}$ in $\mathbb{P}^3$.

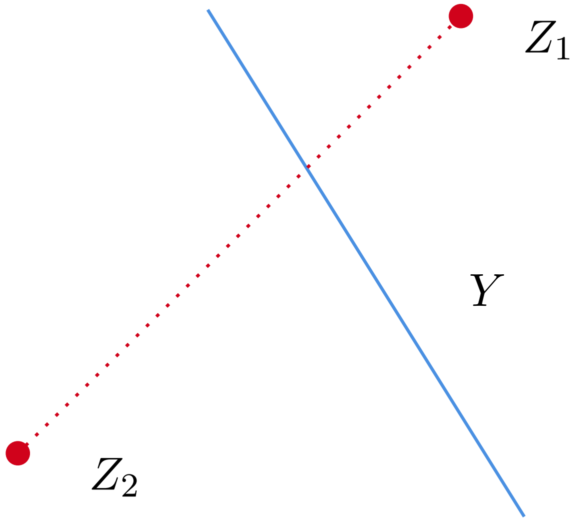


Theorem (Ranestad–Sinn–Telen 24)

The algebraic boundary of the $m = 2$ *amplituhedron is given by* $\langle 12 \rangle, \ldots, \langle n-1\, n \rangle, \langle 1n \rangle = 0$.

Theorem (Even-Zohar–Lakrec–Tessler 25)

The algebraic boundary of the $m = 4$ *amplituhedron is given by* $\langle i\; i+1\; j\; j+1 \rangle = 0$, *for* $1 \leq i < j \leq n$.



The *exterior cyclic polytope* of Z is

$$C_{k,m,n}(Z) := \mathsf{conv}(Z_{i_1} \wedge \ldots \wedge Z_{i_k} \ : \ \{i_1, \ldots, i_k\} \subset [n])$$

$$\text{in } \mathbb{P}(\wedge^k \mathbb{R}^{k+m}).$$

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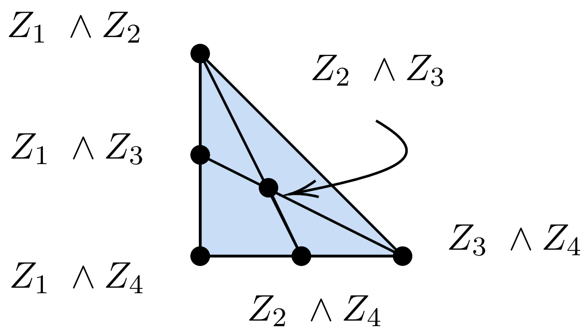
Example (The polytope $C_{2,1,4}(Z)$)

In $(\mathbb{P}^2)^*$, we have



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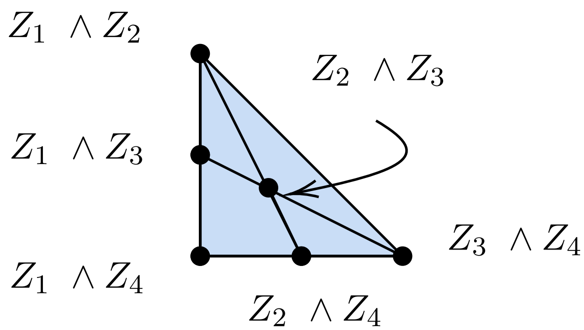

Theorem (Mazzucchelli–P)

The polytope $C_{k,m,n}(Z)$ is the convex hull of $\mathcal{A}_{k,m,n}(Z)$.

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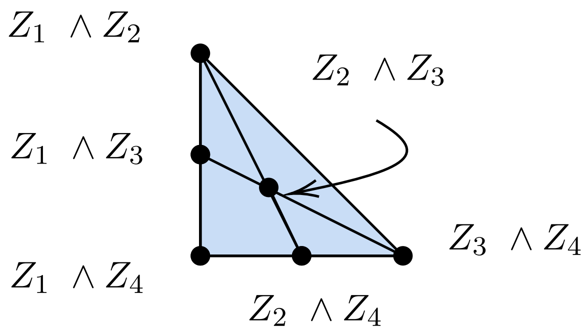
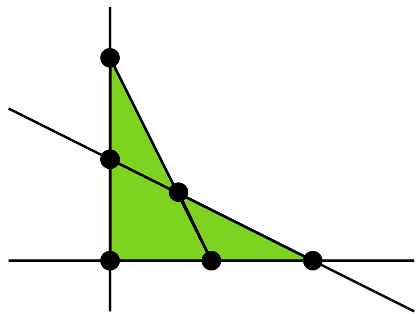


[Karp–Williams 17] The amplituhedron $\mathcal{A}_{2,1,4}(Z)$ looks like



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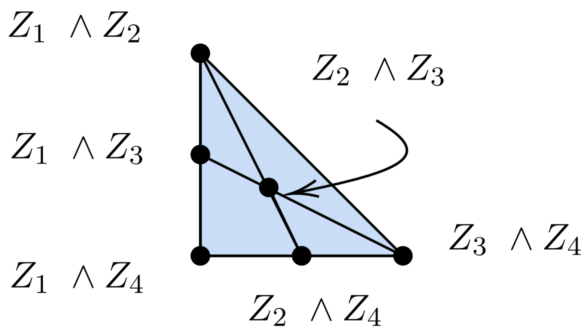

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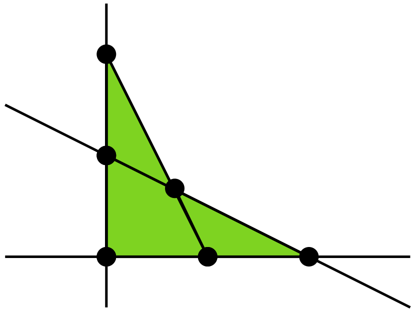

Not convex!

Theorem (Mazzucchelli–P)

The amplituhedron $\mathcal{A}_{2,2,n}(Z)$ *equals* $C_{2,2,n}(Z) \cap$ *Gr*$(2,4)$.

Fix real numbers $0 < a < b < c < d < e < f$ and consider

$$Z \quad = \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & e & f \\ a^2 & b^2 & c^2 & d^2 & e^2 & f^2 \\ a^3 & b^3 & c^3 & d^3 & e^3 & f^3 \end{pmatrix}.$$

Then $C_{2,2,6}(Z)$ is the convex hull in $\mathbb{P}^5$ of the 15 columns of $\wedge^2 Z$:

$$\begin{pmatrix} a-b & a-c & a-d & a-e & \cdots & d-f & e-f \\ a^2-b^2 & a^2-c^2 & a^2-d^2 & a^2-e^2 & \cdots & d^2-f^2 & e^2-f^2 \\ a^3-b^3 & a^3-c^3 & a^3-d^3 & a^3-e^3 & \cdots & d^3-f^3 & e^3-f^3 \\ a^2b-ab^2 & a^2c-ac^2 & a^2d-ad^2 & a^2e-ae^2 & \cdots & d^2f-df^2 & e^2f-ef^2 \\ a^3b-ab^3 & a^3c-ac^3 & a^3d-ad^3 & a^3e-ae^3 & \cdots & d^3f-df^3 & e^3f-ef^3 \\ a^3b^2-a^2b^3 & a^3c^2-a^2c^3 & a^3d^2-a^2d^3 & a^3e^2-a^2e^3 & \cdots & d^3f^2-d^2f^3 & e^3f^2-e^2f^3 \end{pmatrix}.$$

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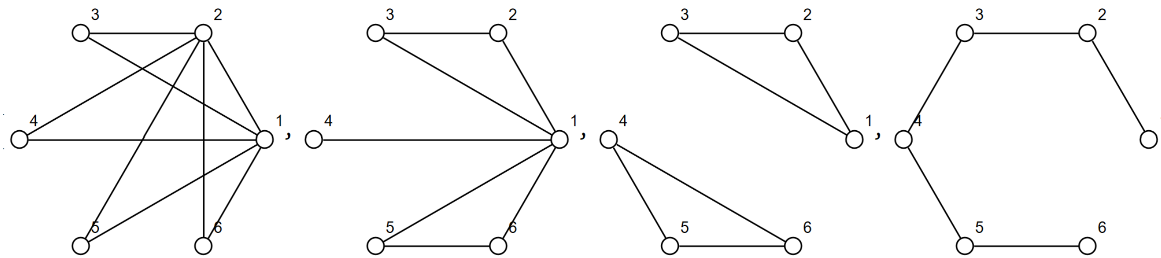
Substituting $(1,3,4,7,8,9)$, it has f-vector $(15,75,143,111,30)$. Among the 30 facets, there are 15 4-simplices, six double pyramids over pentagons, three cyclic polytopes $C(4,6)$, and three with f-vector $(9,26,30,13)$.

Identify vectors $Z_i \wedge Z_j$ with edges ij of a complete graph. There are 30 facets, with four types of supporting hyperplanes:



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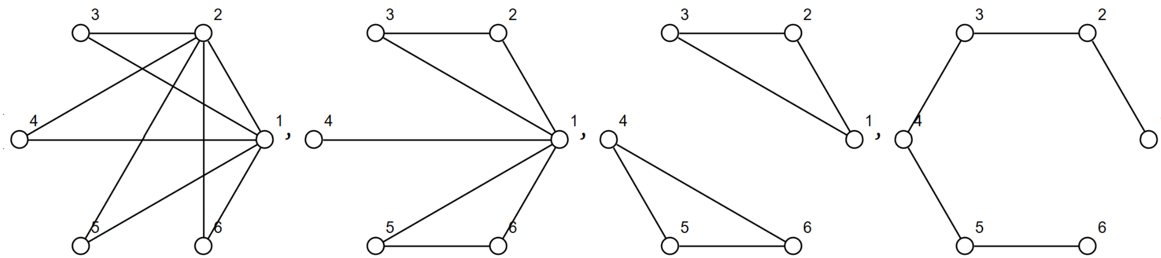


For $(1, 3, 4, 7, 8, f)$, three facets for $f < 45/7$ are

$$\{12, 23, 34, 45, 56\}, \ \{12, 23, 34, 56, 16\}, \ \{12, 16, 34, 45, 56\} .$$

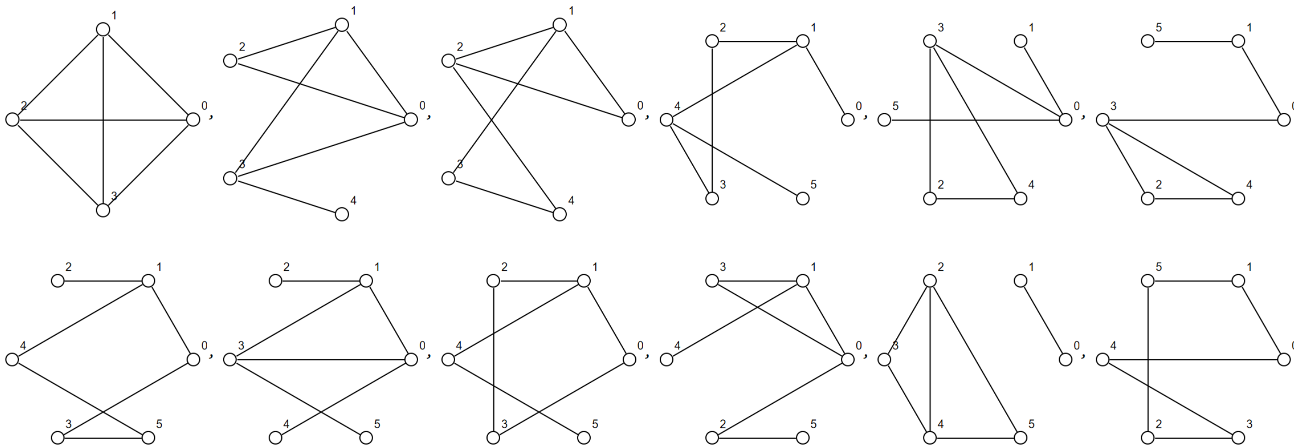
and for $f > 45/7$ change to

$$\{12, 16, 23, 34, 45\}, \ \{12, 16, 23, 45, 56\} \ \{16, 23, 34, 45, 56\} .$$

Combinatorics changes as Z varies over positive matrices!

Of the $\binom{15}{6}$ minors of $\wedge^2 Z$, 1660 are zero and 3345 are nonzero.

Symmetry classes of minors:



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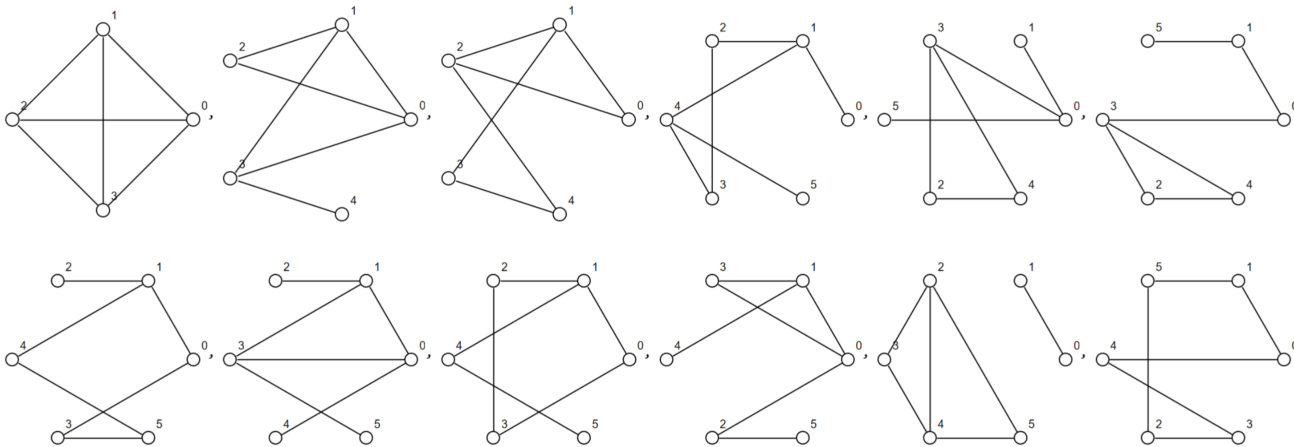

Sign of each minor is fixed by $a < \ldots < f$ except for

$$[12, 23, 34, 45, 56, 16] =$$
$$(a-c)(a-d)(a-e)(b-d)(b-e)(b-f)(d-f)(c-e)(c-f)$$
$$\cdot\,(abd - abe - acd + acf + ade - adf + bce - bcf - bde + bef + cdf - cef)\,.$$

Theorem (Mazzucchelli–P)

The combinatorial type of $C_{2,2,n}(Z)$ *is constant for positive* $4 \times n$ *matrices* Z *outside the closed locus where the polynomial* $\det[Z_1 \wedge Z_2 \ \ \ldots \ \ Z_5 \wedge Z_6 \ \ Z_6 \wedge Z_1]$ *or one of its permutations is zero.*

In Plücker coordinates on $Z \in \mathsf{Gr}(4, n)$:

$$p_{1234}p_{1356}p_{2456} - p_{1235}p_{1346}p_{2456} + p_{1235}p_{1246}p_{3456} \,.$$

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For $k = m = 2$, small f-vectors include:

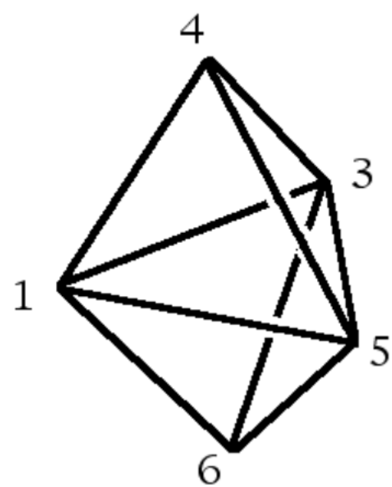
$n = 5$:	10	35	55	40	12	1
$n = 6$:	15	75	143	111	30	1
$n = 7$:	21	147	328	282	82	1
$n = 8$:	28	266	664	616	192	1
$n = 9$:	36	450	1217	1191	390	1

What is a *dual amplituhedron*?

Andrew Hodges, *Eliminating spurious poles from gauge-theoretic amplitudes* (2009):

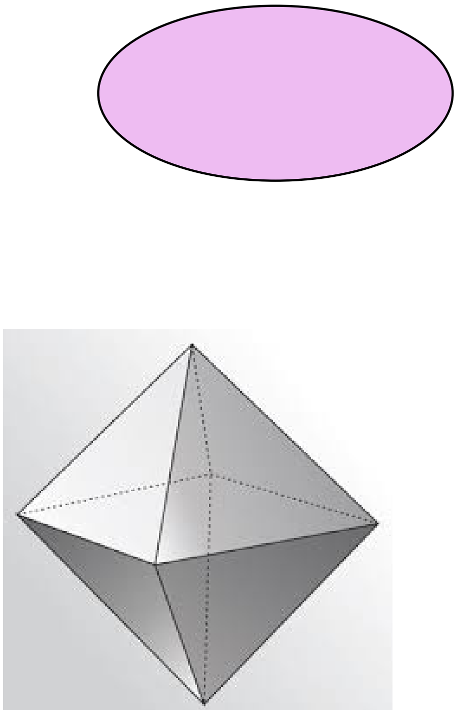
$$A(1^-2^-3^-4^+5^+) = \frac{[45]^4}{[12][23][34][45][51]} = \frac{\langle 12\rangle^4\langle 23\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\int_{P_5}(W.Z_2)^{-4}DW\,.$$

Here P_5 is the dual of

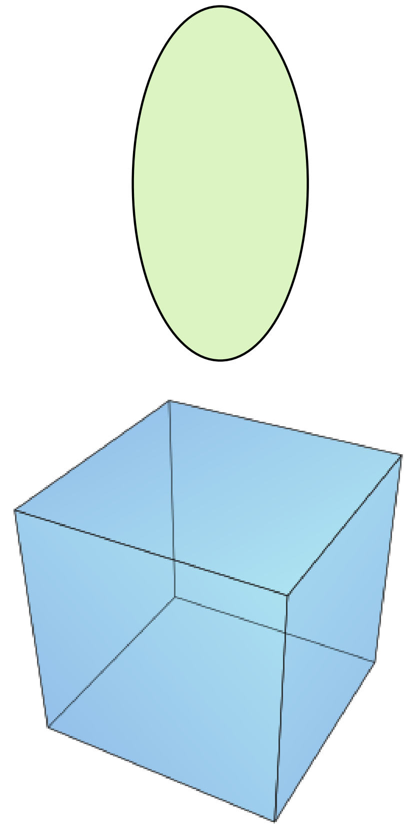


The *polar dual* of a semialgebraic set $S \subset \mathbb{R}^n$ is

$$S^* := \{ l \in (\mathbb{R}^n)^* \, : \, l(x) \geq -1 \; \forall \, x \in S \} \; .$$



S



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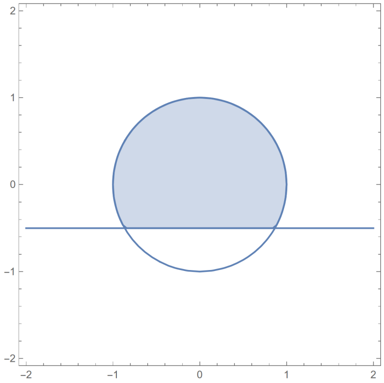

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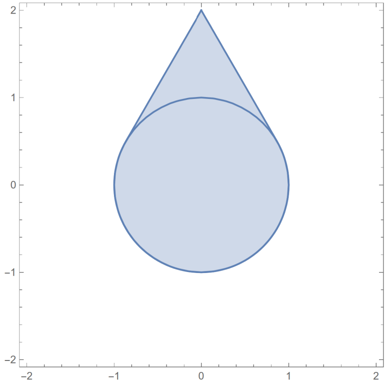

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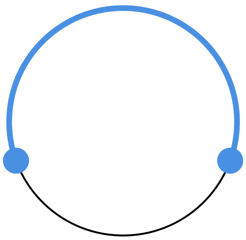

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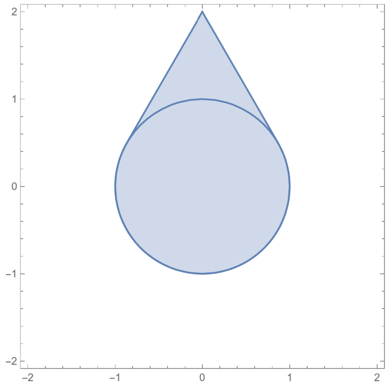

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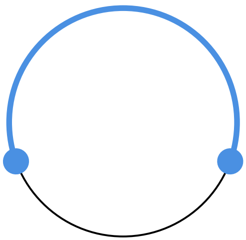

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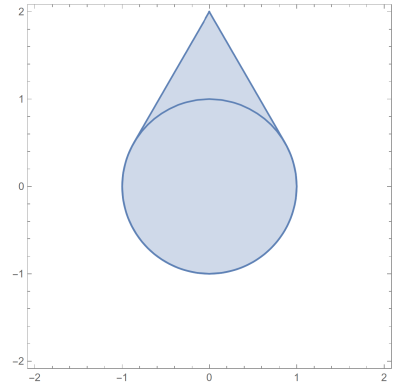

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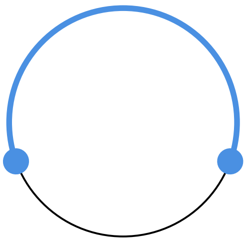

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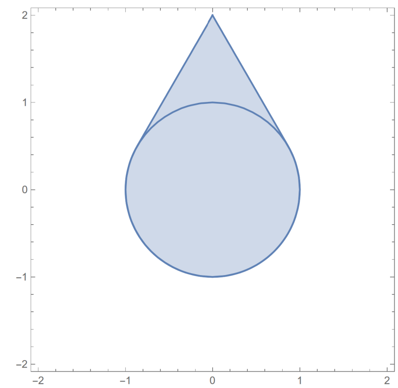

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The *extendable dual amplituhedron* is

$$\mathcal{A}^*_{k,m,n} := Gr(m, k+m) \cap \mathrm{conv}(\mathcal{A}_{k,m,n})^* = Gr(m, k+m) \cap C^*_{k,m,n}.$$

Define

$$W_i := Z_{i-m+1} \wedge Z_{i-m+2} \wedge \cdots \wedge Z_i \wedge \cdots \wedge Z_{i+k-1}\,, \qquad i \in [n]\,.$$

The *twist map* is

$$\begin{aligned} \tau \;:\; \mathsf{Mat}_{>0}(k+m,n) &\to \mathsf{Mat}_{>0}(k+m,n)\,, \\ Z &\mapsto W\,, \end{aligned}$$

where W has columns $W_1, \ldots, W_n$. [Marsh–Scott 13]

Example

$[Z_1 \ \ldots \ Z_6] \mapsto [Z_6 \wedge Z_1 \wedge Z_2 \quad Z_1 \wedge Z_2 \wedge Z_3 \quad \ldots \quad Z_5 \wedge Z_6 \wedge Z_1]$.

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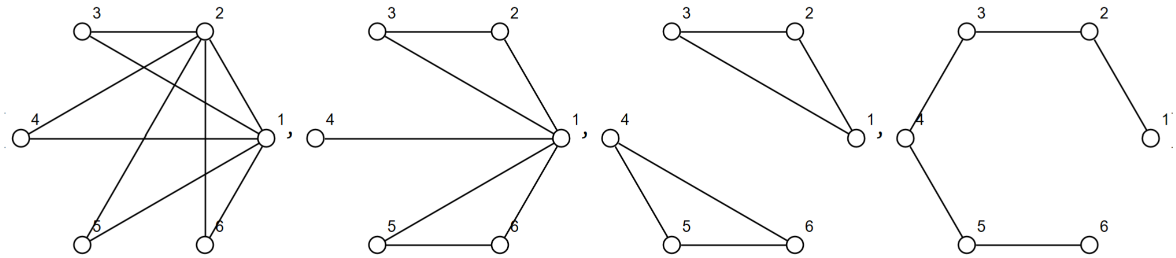
Theorem (Mazzucchelli–P)

There is an equality

$$\mathcal{A}_{2,2,n}(Z)^* = \mathcal{A}_{2,2,n}(W).$$

$\mathcal{A}_{2,2,n}(Z)^*$ is an amplituhedron for another particle configuration!

For $C_{2,2,6}(Z)$ there are four types of supporting hyperplanes:



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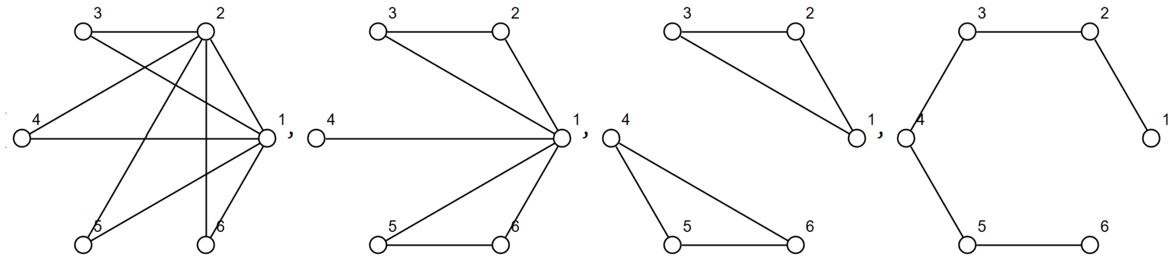

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- lines meeting (12) in $\mathbb{P}^3$ $\leftarrow$ defining equation $\langle 12 \rangle = 0$
- lines meeting $(123) \cap (156)$ in $\mathbb{P}^3$
- lines meeting $(123) \cap (456)$ in $\mathbb{P}^3$

Theorem (Mazzucchelli-P)

The supporting Schubert hyperplanes of $C_{2,2,n}(Z)$ are exactly the $\binom{n}{2}$ hyperplanes consisting of lines meeting $(i-1\, i\, i+1) \cap (j-1\, j\, j+1)$ for $1 \leq i < j \leq n$. Furthermore, they intersect transversally in Gr$(2,4)$ for every $Z \in \mathrm{Mat}_{>0}(4, \mathrm{n})$.

The *Schubert exterior cyclic polytope* $\widetilde{C}_{k,m,n}(Z)$ is obtained from $C_{k,m,n}(Z)$ by deleting all facet inequalities whose supporting hyperplanes are not Schubert divisors.

Proposition (Mazzucchelli–P)

There is an equality

$$\widetilde{C}_{2,2,n}(Z) = C_{2,2,n}(W)^*.$$

Example

The f-vector of $C_{2,2,6}$ is

$$(15, 75, 143, 111, 30).$$

The f-vector of $\widetilde{C}_{2,2,6}$ is

$$(30, 111, 143, 75, 15).$$

Thank you for listening!

The wedge power matroid

The *wedge power matroid* $W_{k,m,n}$ is the matroid of the point configuration $Z_{i_1} \wedge \ldots \wedge Z_{i_k}$, for Z generic*.

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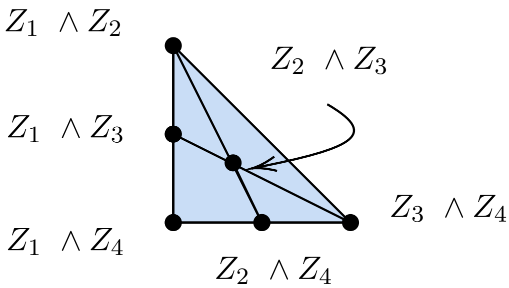

We have

$$aZ_2+bZ_3+cZ_4 = Z_1 \implies aZ_1\wedge Z_2+bZ_1\wedge Z_3+cZ_1\wedge Z_4 = Z_1\wedge Z_1 = 0.$$

Non-bases are $\{12,13,14\},\{12,23,24\},\{13,23,34\},\{14,24,34\}$.

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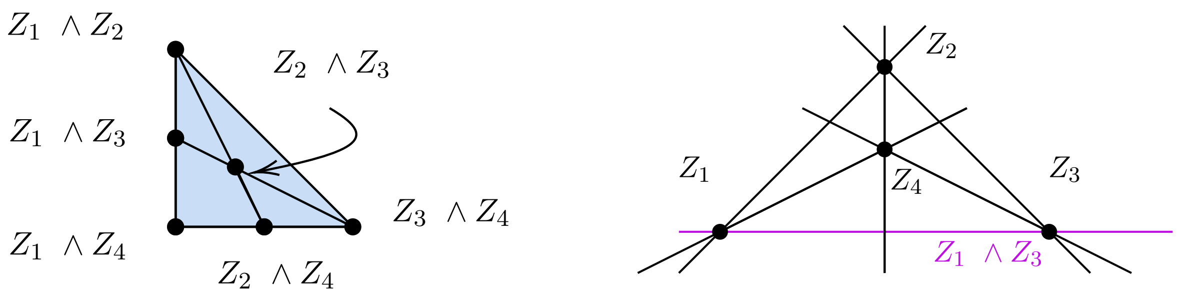

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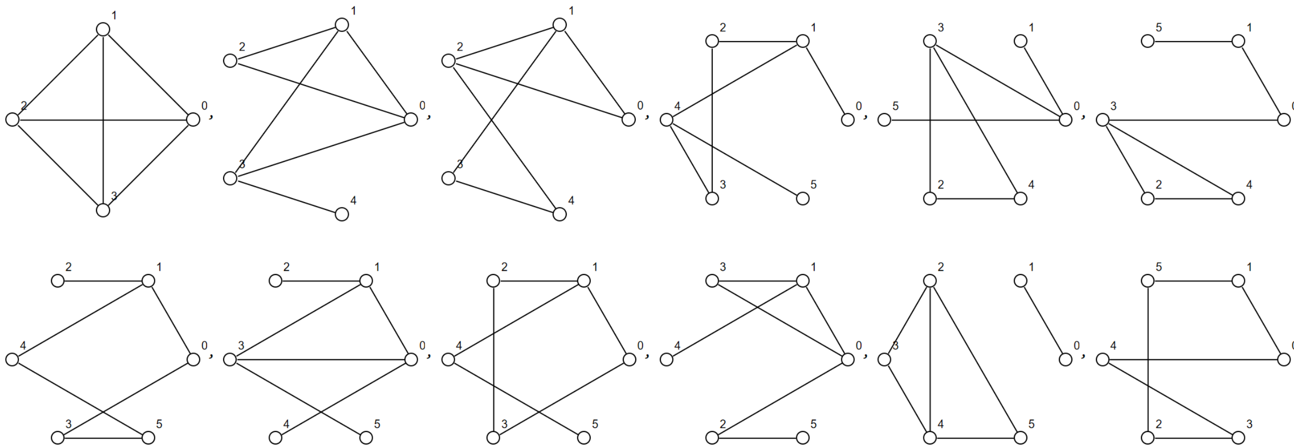
Remark

The matroid $W_{k,1,k+1}$ is the matroid of the *braid arrangement.*

Of the $\binom{15}{6}$ minors of $\wedge^2 Z$, 1660 are zero and 3345 are nonzero.

Symmetry classes of minors:



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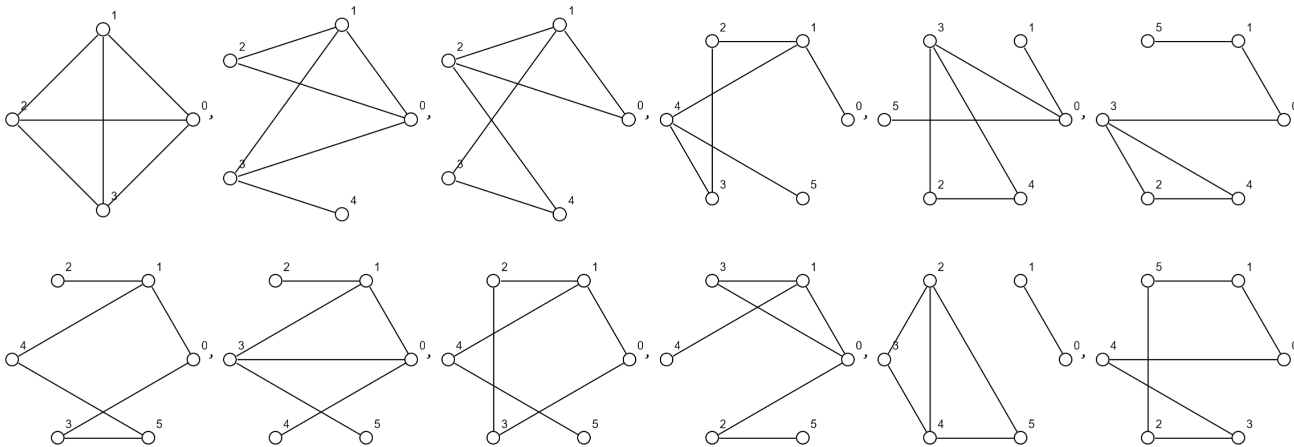


Sign of each minor is fixed by $a < \ldots < f$ except for

$$[12, 23, 34, 45, 56, 16] =$$
$$(a-c)(a-d)(a-e)(b-d)(b-e)(b-f)(d-f)(c-e)(c-f)$$
$$\cdot\,(abd - abe - acd + acf + ade - adf + bce - bcf - bde + bef + cdf - cef)\,.$$

The wedge power matroid $W_{k,m,n}$

The case $m = 1$:

- Matroid of discriminantal arrangement of n general points in $\mathbb{P}^k$ [Manin–Schechtman 89]

The case $k = 2$:

- Dual of Kalai's *hyperconnectivity matroid* $\mathcal{H}_{n-m-2}(n)$ [Kalai 85, Brakensiek–Dhar–Gao–Gopi–Larson 24]
- $\mathcal{H}_d(n)$ is the algebraic matroid of $n \times n$ skew-symmetric matrices of rank at most d [Ruiz–Santos 23]

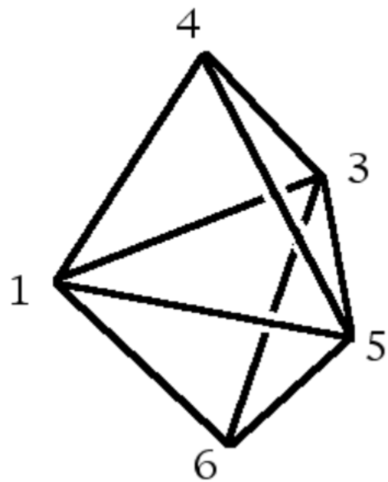
The case $k = 2$ and $n = m + 4$:

- Graphical characterization of bases of $\mathcal{H}_2(n)$ [Bernstein 17]
- $\mathcal{H}_2(n)$ is the algebraic matroid of Gr$(2, n)$

Upshot: describing bases of $W_{k,m,n}$ and faces of $C_{k,m,n}(Z)$ is hard!

Andrew Hodges, *Eliminating spurious poles from gauge-theoretic amplitudes* (2009):

$$A(1^-2^-3^-4^+5^+) = \frac{[45]^4}{[12][23][34][45][51]} = \frac{\langle 12\rangle^4\langle 23\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\int_{P_5}(W.Z_2)^{-4}DW\,.$$



What is a *dual amplituhedron*?