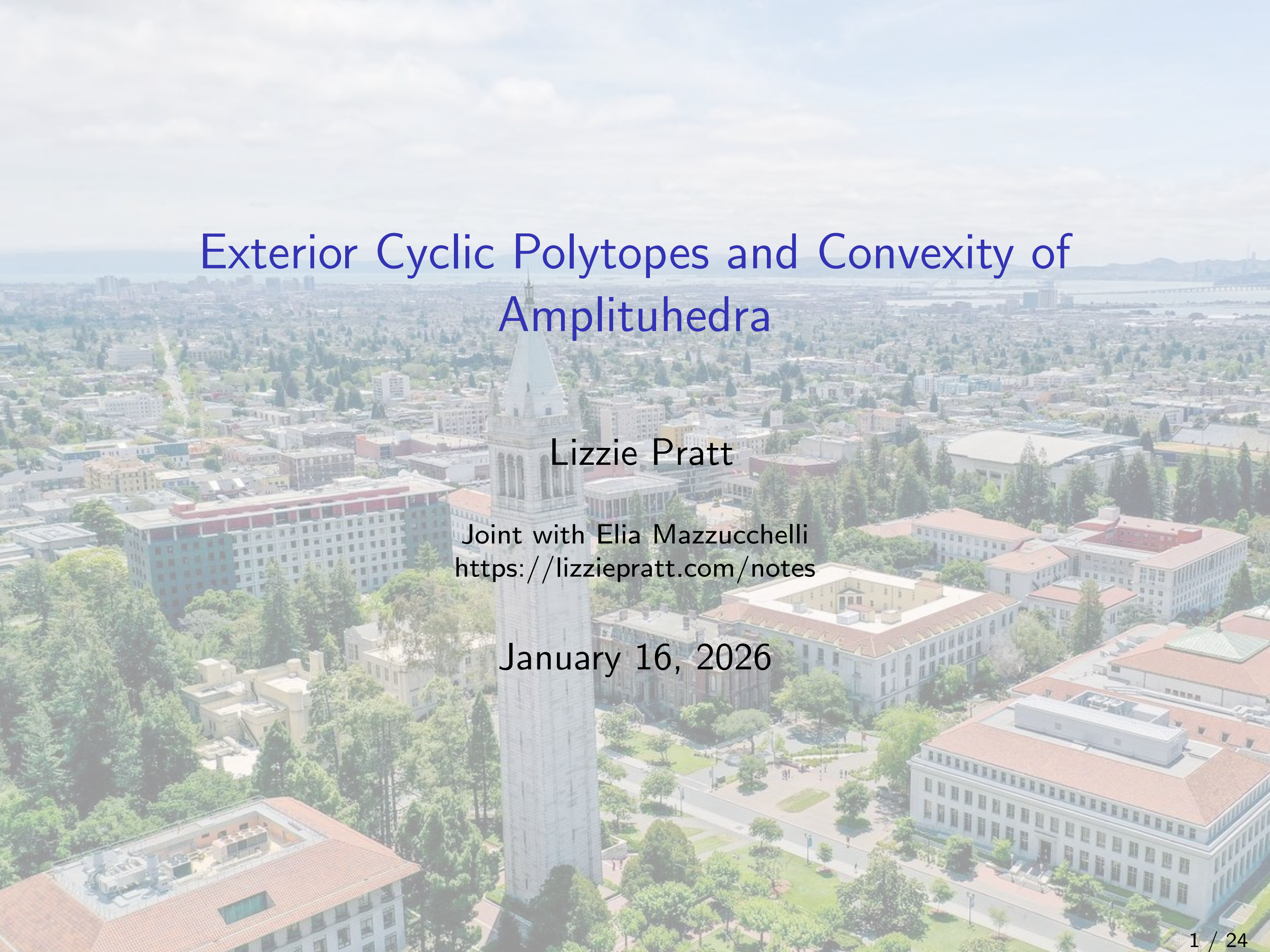


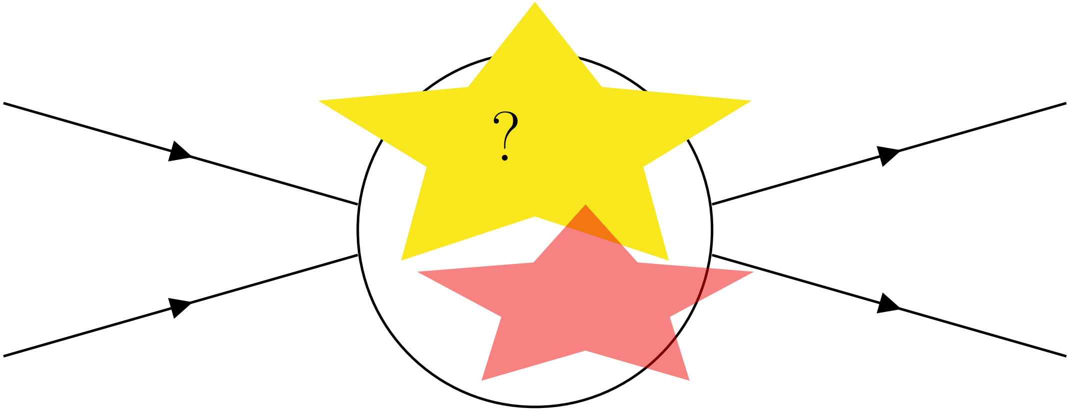
Exterior Cyclic Polytopes and Convexity of Amplituhedra

Lizzie Pratt

Joint with Elia Mazzucchelli
https://lizziepratt.com/notes

January 16, 2026

Goal: predict outcome of particle collisions scattering amplitude.



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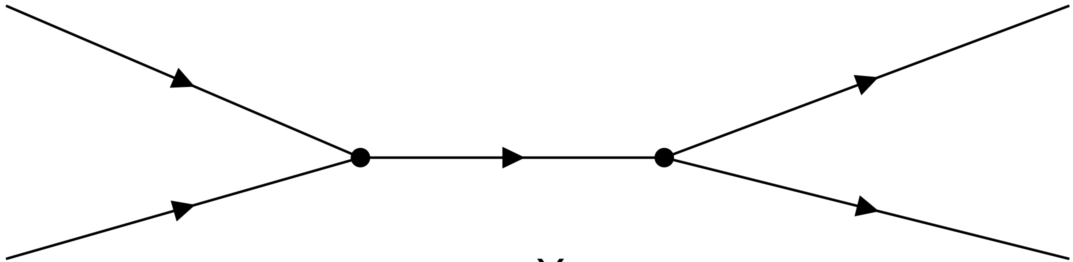
Classically:



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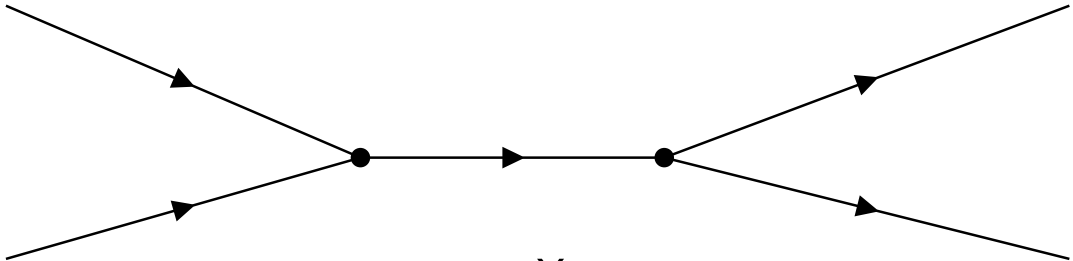
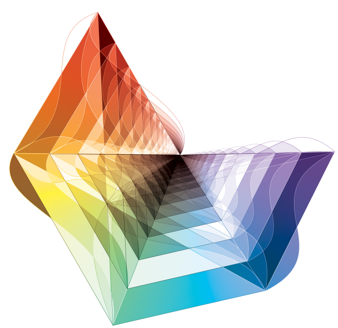


$$A = \sum_{G} I_G$$

Arkani-Hamed and Trnka, *The Amplituhedron* (2013): amplitudes in tree-level $N = 4$ super Yang-Mills have poles along the boundaries of certain semialgebraic sets!



Semialgebraic sets in projective space

- A *basic semialgebraic cone* in R^{n+1} is a set defined by homogeneous equations and inequalities
- A *semialgebraic set* S $\quad P^n$ is the projection of a semialgebraic cone in R^{n+1} under

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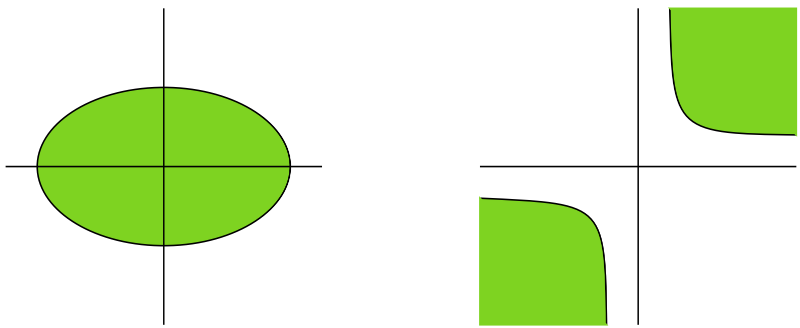

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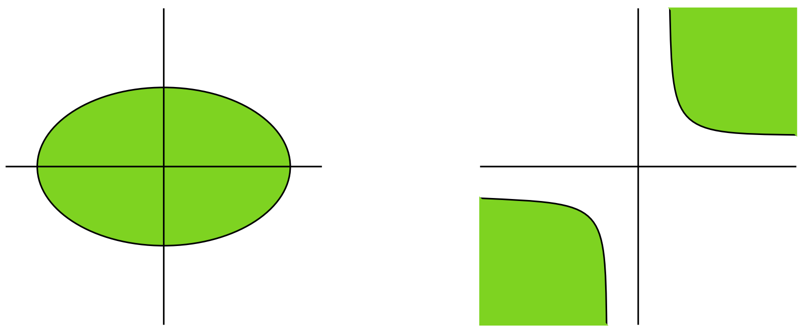


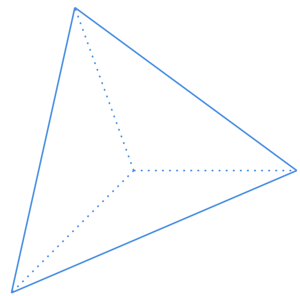
Eg $xz \quad y^2 \quad 0$

Theorem (Kummer–Sinn 22)

The convex hull of a connected set $S \quad \mathbb{P}^n$ *may be computed in any affine chart fully containing* S*.*

The *projective simplex* is

$$\Delta_n := \mathbb{P}\mathrm{conv}\{e_0, \ldots, e_n\} \subseteq \mathbb{P}^n.$$



The *Grassmannian* parameterizes k-spaces in $\mathbb{R}^n$, and is a projective variety via

$$\begin{aligned} \mathrm{Gr}(k,n) &\to \mathbb{P}(\wedge^k \mathbb{R}^n) \\ \mathrm{span}(v_1, \ldots, v_k) &\mapsto v_1 \wedge \cdots \wedge v_k. \end{aligned}$$

The *positive Grassmannian* is

$$\mathrm{Gr}_{\geq 0}(k,n) := \Delta_{\binom{n}{k}-1} \cap \mathrm{Gr}(k,n).$$

Let Z be a $(k+m) \times n$ matrix with positive maximal minors.

$$\wedge^k Z : Gr(k,n) \dashrightarrow Gr(k,k+m)$$
$$\mathrm{span}(v_1,\ldots,v_k) \mapsto \mathrm{span}(Zv_1,\ldots,Zv_k).$$

The *amplituhedron* $\mathcal{A}_{k,m,n}(Z)$ is the image of $\mathrm{Gr}_{\geq 0}(k,n)$.

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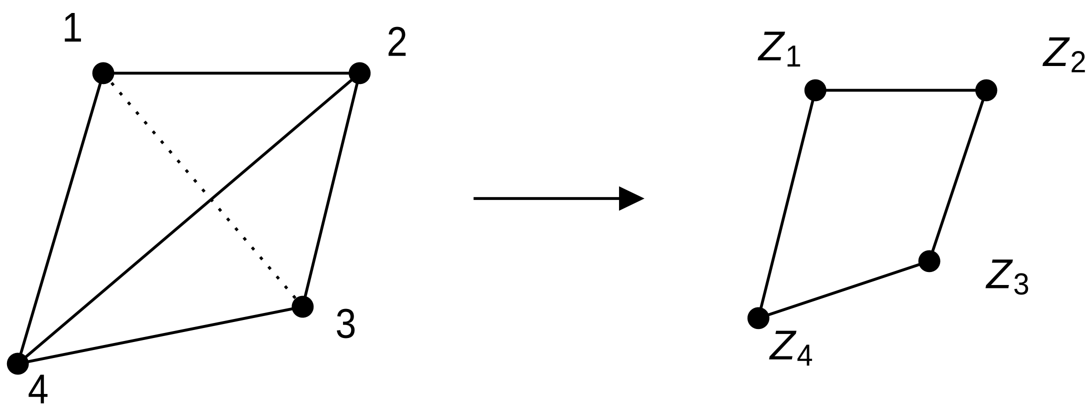
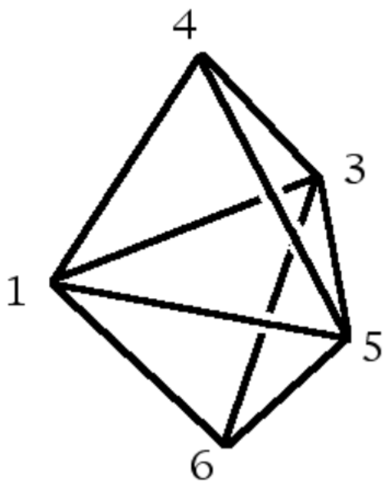
Example ($k=1$)

$$Z : \Delta_{n-1} \to \mathbb{P}^m$$
$$e_i \mapsto Z_i$$



The image is a *cyclic polytope*.

Some cyclic polytopes in $\mathbb{P}^3$:



[Hodges 2009]

Gr $_{0}(k;n)$: linear (simplex) \ nonlinear (Grassmannian).
What about $A_{k;m;n}$??

$\mathrm{Gr}_{\;0}(k;n)$: linear (simplex) $\setminus$ nonlinear (Grassmannian).
What about $A_{k;m;n}$??

The *twistor coordinates* wrt Z on $\mathrm{Gr}(k, k+2)$ are

$$\langle i j \rangle := \det[Z_i \, Z_j \, Y^T], \qquad [Y] \in \mathrm{Gr}(k, k+2).$$

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On $\mathrm{Gr}(2, 4)$, we have

$$\langle 12 \rangle = (z_{1i}z_{2j} - z_{2i}z_{1j})p_{34} - (z_{1i}z_{3j} - z_{3i}z_{1j})p_{24} + (z_{2i}z_{3j} - z_{3i}z_{2j})p_{14} + \dots$$

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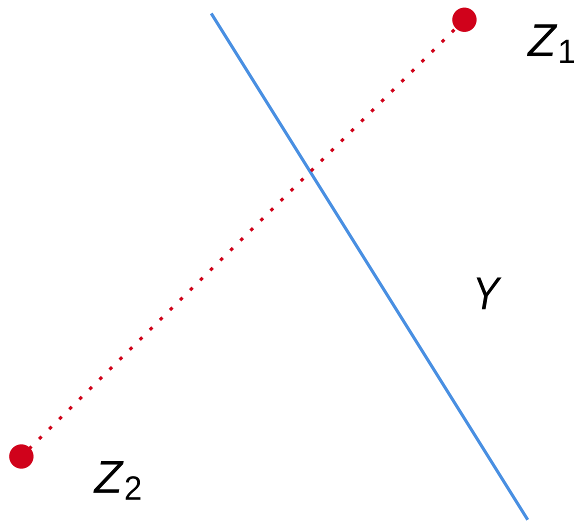
This vanishes on lines $[Y]$ meeting the line $\overline{Z_1 Z_2}$ in $\mathbb{P}^3.$

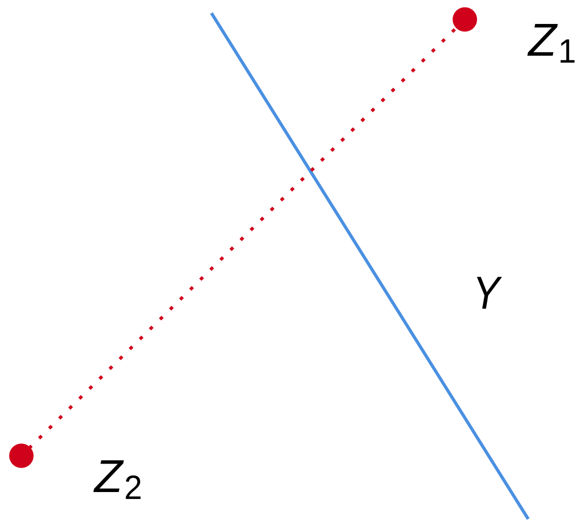


Theorem (Ranestad–Sinn–Telen 24)

The algebraic boundary of the $m = 2$ amplituhedron is given by
$$\langle 12\rangle, \dots, \langle n-1\; n\rangle, \langle 1n\rangle = 0.$$

Theorem (Even-Zohar–Lakrec–Tessler 25)

The algebraic boundary of the $m = 4$ amplituhedron is given by
$$\langle i\; i+1\; j\; j+1\rangle = 0, \text{ for } 1 \leq i < j \leq n.$$



The *exterior cyclic polytope* of Z is

$$C_{k,m,n}(Z) := \mathrm{Pconv}(Z_{i_1} \wedge \ldots \wedge Z_{i_k} \;:\; \{i_1,\ldots,i_k\} \subseteq [n]) \quad \text{in } \mathbb{P}(\wedge^k \mathbb{R}^{k+m}).$$

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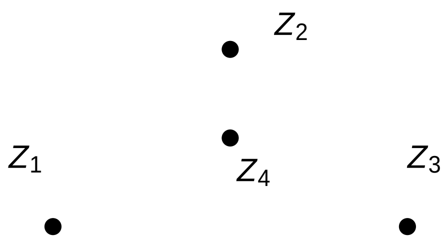
Example (The polytope $C_{2,1,4}(Z)$)



In $\mathbb{P}^2$

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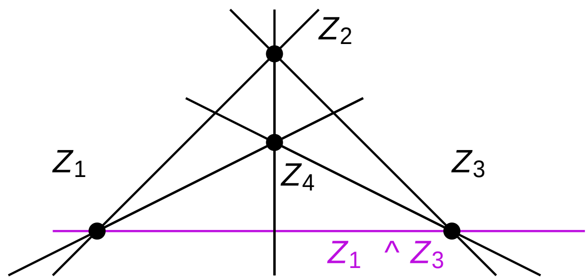

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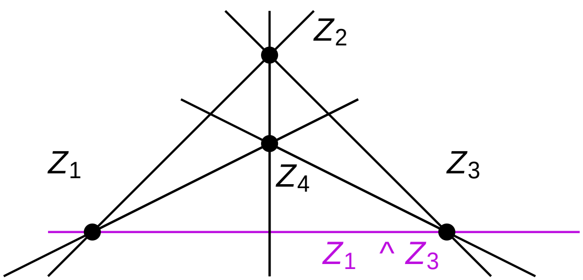

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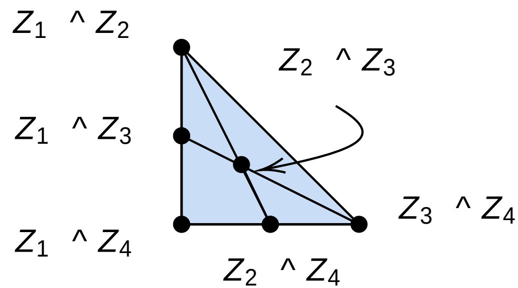
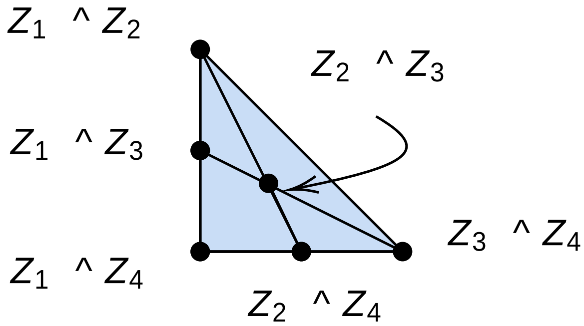


In $(\mathbb{P}^2)^*$

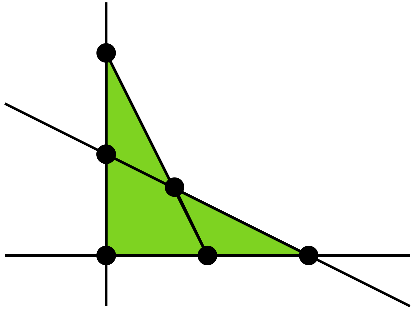
Theorem (Mazzucchelli–P)

The polytope $C_{k,m,n}(Z)$ is the convex hull of $A_{k,m,n}(Z)$.

The polytope $C_{2;1;4}(Z)$ looks like



[Karp–Williams 17] The amplituhedron $A_{2;1;4}(Z)$ looks like



Not convex!

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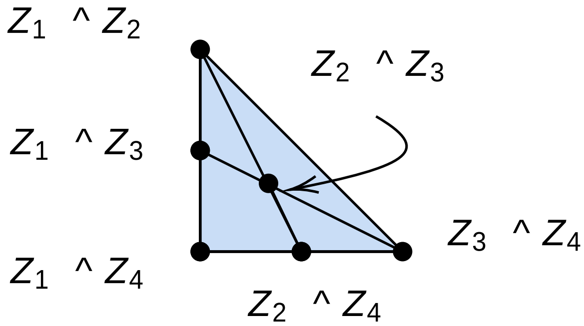

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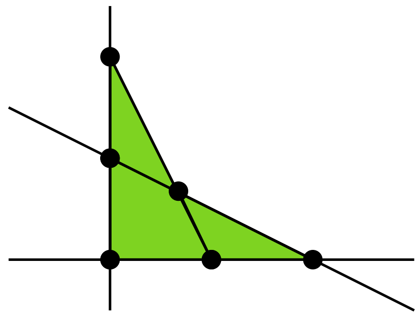

Not convex!

Theorem (Mazzucchelli–P)

The amplituhedron $A_{2;2;n}(Z)$ *equals* $C_{2;2;n}(Z) \setminus Gr(2;4)$*.*

Fix real numbers $0 < a < b < c < d < e < f$ and consider

$$Z \quad = \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & e & f \\ a^2 & b^2 & c^2 & d^2 & e^2 & f^2 \\ a^3 & b^3 & c^3 & d^3 & e^3 & f^3 \end{pmatrix} .$$

Then $C_{2,2,6}(Z)$ is the convex hull in $\mathbb{P}^5$ of the 15 columns of $\wedge^2 Z$:

$$\begin{pmatrix} a & b & a & c & a & d & a & e & & d & f & e & f \\ a^2 & b^2 & a^2 & c^2 & a^2 & d^2 & a^2 & e^2 & & d^2 & f^2 & e^2 & f^2 \\ a^3 & b^3 & a^3 & c^3 & a^3 & d^3 & a^3 & e^3 & & d^3 & f^3 & e^3 & f^3 \\ a^2 b & ab^2 & a^2 c & ac^2 & a^2 d & ad^2 & a^2 e & ae^2 & & d^2 f & df^2 & e^2 f & ef^2 \\ a^3 b & ab^3 & a^3 c & ac^3 & a^3 d & ad^3 & a^3 e & ae^3 & & d^3 f & df^3 & e^3 f & ef^3 \\ a^3 b^2 & a^2 b^3 & a^3 c^2 & a^2 c^3 & a^3 d^2 & a^2 d^3 & a^3 e^2 & a^2 e^3 & & d^3 f^2 & d^2 f^3 & e^3 f^2 & e^2 f^3 \end{pmatrix} .$$

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Substituting $(1;3;4;7;8;9)$; it has f-vector $(15;75;143;111;30)$. Among the 30 facets, there are 15 4-simplices, six double pyramids over pentagons, three cyclic polytopes $C(4;6)$; and three with f-vector $(9;26;30;13)$:

Identify vectors $Z_i \wedge Z_j$ with edges ij of a complete graph. There are 30 facets, with four types of supporting hyperplanes:

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For (1; 3; 4; 7; 8; f); three facets for f < 45=7 are

f12; 23; 34; 45; 56g; f12; 23; 34; 56; 16g; f12; 16; 34; 45; 56g :

and for f > 45=7 change to

f12; 16; 23; 34; 45g; f12; 16; 23; 45; 56g f16; 23; 34; 45; 56g :

Combinatorics changes as Z varies over positive matrices!

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For $(1, 3, 4, 7, 8, f)$, three facets for $f < 45/7$ are

$$\{12, 23, 34, 45, 56\}, \quad \{12, 23, 34, 56, 16\}, \quad \{12, 16, 34, 45, 56\}.$$

and for $f > 45/7$ change to

$$\{12, 16, 23, 34, 45\}, \quad \{12, 16, 23, 45, 56\} \quad \{16, 23, 34, 45, 56\}.$$

Combinatorics changes as Z varies over positive matrices! This is because the oriented matroid of $\wedge^k Z$ changes.

The wedge power matroid

The wedge power matroid $W_{k;m;n}$ is the matroid of the point configuration $Z_{i_1} \wedge \dots \wedge Z_{i_k}$, for Z generic*.

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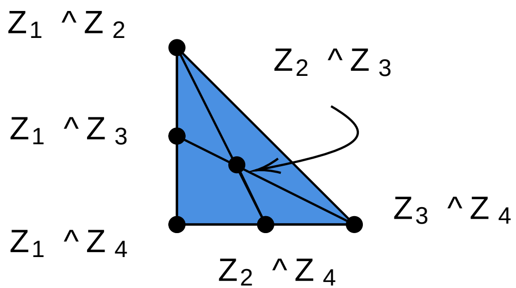
Example



Non-bases are $\{12; 13; 14\}; \{12; 23; 24\}; \{13; 23; 34\}; \{14; 24; 34\}.$

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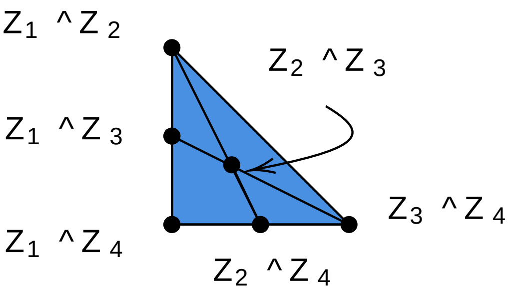

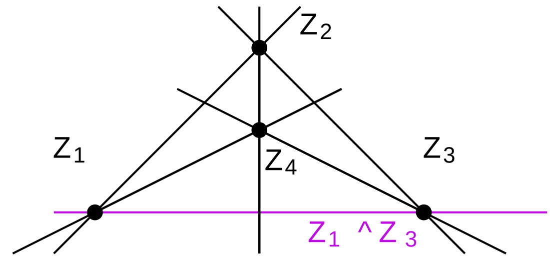

Non-bases are $\{12, 13, 14\}, \{12, 23, 24\}, \{13, 23, 34\}, \{14, 24, 34\}$.

Remark

The matroid $W_{k;1;k+1}$ is the matroid of the braid arrangement.

The wedge power matroid $W_{k,m,n}$

The case $m = 1$:

- Matroid of discriminantal arrangement of n general points in $\mathbb{P}^k$ [Manin{Schechtman 89]

The case $k = 2$:

- Dual of Kalai's hyperconnectivity matroid $H_{n-m-2}(n)$ [Kalai 85, Brakensiek{Dhar{Gao{Gopi{Larson 24]
- $H_d(n)$ is the algebraic matroid of $n \times n$ skew-symmetric matrices of rank at most d [Ruiz{Santos 23]

The case $k = 2$ and $n = m + 4$:

- Graphical characterization of bases of $H_2(n)$ [Bernstein 17]
- $H_2(n)$ is the algebraic matroid of $Gr(2, n)$

Upshot: describing bases of $W_{k,m,n}$ and faces of $C_{k,m,n}(Z)$ is hard!

Of the $\binom{15}{6}$ minors of $\wedge^2 Z$; 1660 are zero (nonbases) and 3345 are nonzero (bases).

Symmetry classes of minors:

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Symmetry classes of minors:

Sign of each minor is xed by $a < \ldots < f$ except for

$$[12; 23; 34; 45; 56; 16] =$$

$$(a\ c)(a\ d)(a\ e)(b\ d)(b\ e)(b\ f)(d\ f)(c\quad e)(c\quad f)$$

$$(abd\quad abe\quad acd + acf + ade\quad adf + bce\quad bcf\quad bde + bef + cdf\quad cef):$$

Theorem (Mazzucchelli–P)

The combinatorial type of $C_{2,2,n}(Z)$ is constant for positive $4 \quad n$ matrices Z outside the closed locus where the polynomial $\det[Z_1 \wedge Z_2 \; \dots \; Z_5 \wedge Z_6 \quad Z_6 \wedge Z_1]$ or one of its permutations is zero.

In Plücker coordinates on $Z \in Gr(4, n)$:

$$p_{1234}p_{1356}p_{2456} \quad p_{1235}p_{1346}p_{2456} + p_{1235}p_{1246}p_{3456}.$$

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For $k = m = 2$, small f-vectors include:

$n = 5$	:	10	35	55	40	12	1
$n = 6$	:	15	75	143	111	30	1
$n = 7$	:	21	147	328	282	82	1
$n = 8$	:	28	266	664	616	192	1
$n = 9$	:	36	450	1217	1191	390	1

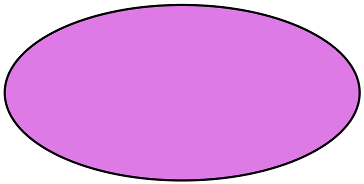
What is a dual amplituhedron?

Andrew Hodges, Eliminating spurious poles from gauge-theoretic amplitudes (2009):

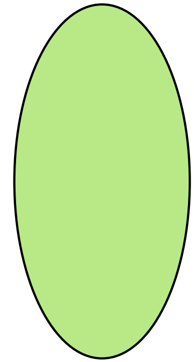
Here P_5 is the dual of

The polar dual of a semialgebraic set S R^n is

$$S \;:= fy\ 2\ R^n : x \;\; y \quad 1\ 8x\ 2\ Sg :$$



S



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The polar dual of a semialgebraic set $S \quad \mathbb{R}^n$ is

$$S \quad := \{y \in \mathbb{R}^n : x \quad y \quad 1 \ \forall x \in S\} :$$

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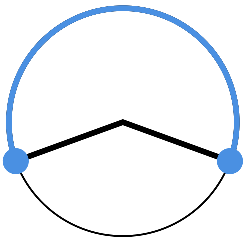
S S

The polar dual of a semialgebraic set $S \subset \mathbb{R}^n$ is

$$S^\circ := \{y \in \mathbb{R}^n : x \cdot y \le 1 \ \forall x \in S\}.$$



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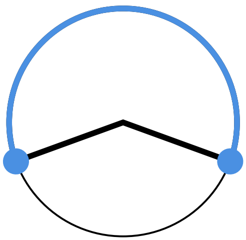
The polar dual of a semialgebraic set $S \subseteq \mathbb{R}^n$ is

$$S := \{y \in \mathbb{R}^n : x \; y \quad 1 \; \forall x \in S\} :$$



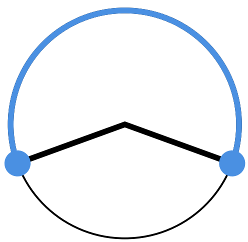
S

S

Observation: $S = \mathrm{conv}(S)$: Very big!

The polar dual of a semialgebraic set $S \subseteq \mathbb{R}^n$ is

$$S^{\circ} := \{y \in \mathbb{R}^n : x \cdot y \geq 1 \ \forall x \in S\}.$$



S

S

Observation: $S^{\circ} = \mathrm{conv}(S)^{\circ}$: Very big!
The extendable dual amplituhedron is

$$A_{k,m,n} := \mathrm{Gr}(m, k+m) \cap C_{k,m,n}.$$

Define

$$W_i := Z_{i-m+1} \wedge Z_{i-m+2} \wedge \cdots \wedge Z_i \wedge \cdots \wedge Z_{i+k-1}\,, \qquad i \in [n].$$

The twist map is

$$\tau : \mathrm{Mat}_{>0}(k+m; n) \to \mathrm{Mat}_{>0}(k+m; n)\,,$$
$$Z \mapsto W\,,$$

where W has columns $W_1, \ldots, W_n$. [Marsh–Scott 13]

Example

$$[Z_1 \cdots Z_6] \mapsto [Z_6 \wedge Z_1 \wedge Z_2 \quad Z_1 \wedge Z_2 \wedge Z_3 \quad \cdots \quad Z_5 \wedge Z_6 \wedge Z_1].$$

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Theorem (Mazzucchelli–P)

There is an equality

$$\mathcal{A}_{2,2,n}(Z) = \mathcal{A}_{2,2,n}(W).$$

$\mathcal{A}_{2,2,n}(Z)$ is an amplituhedron for another particle configuration!

For $C_{2;2;6}(Z)$ there are four types of supporting hyperplanes:

The rst three come from Schubert divisors, which consist of

- lines meeting (12) in $\mathbb{P}^3$

For $C_{2,2,6}(Z)$ there are four types of supporting hyperplanes:

The first three come from Schubert divisors, which consist of

- lines meeting (12) in $\mathbb{P}^3$ defining equation $\langle 12 \rangle = 0$
- lines meeting $(123) \cap (156)$ in $\mathbb{P}^3$
- lines meeting $(123) \cap (456)$ in $\mathbb{P}^3$

Theorem (Mazzucchelli-P)

The supporting Schubert hyperplanes of $C_{2,2,n}(Z)$ are exactly the $\binom{n}{2}$ hyperplanes consisting of lines meeting $(i-1\ i\ i+1) \cap (j-1\ j\ j+1)$ for $1 \leq i < j \leq n$. Furthermore, they intersect transversally in $Gr(2,4)$ for every $Z \in Mat_{>0}(4,n)$.

The Schubert exterior cyclic polytope $\widetilde{C}_{k;m;n}(Z)$ is obtained from $C_{k;m;n}(Z)$ by deleting all facet inequalities whose supporting hyperplanes are not Schubert divisors.

Proposition (Mazzucchelli–P)

There is an equality

$$\widetilde{C}_{2;2;n}(Z) = C_{2;2;n}(W) \; .$$

Example

The f-vector of $C_{2;2;6}$ is

$$(15; 75; 143; 111; 30).$$

The f-vector of $\widetilde{C}_{2;2;6}$ is

$$(30; 111; 143; 75; 15).$$

Thank you for listening!