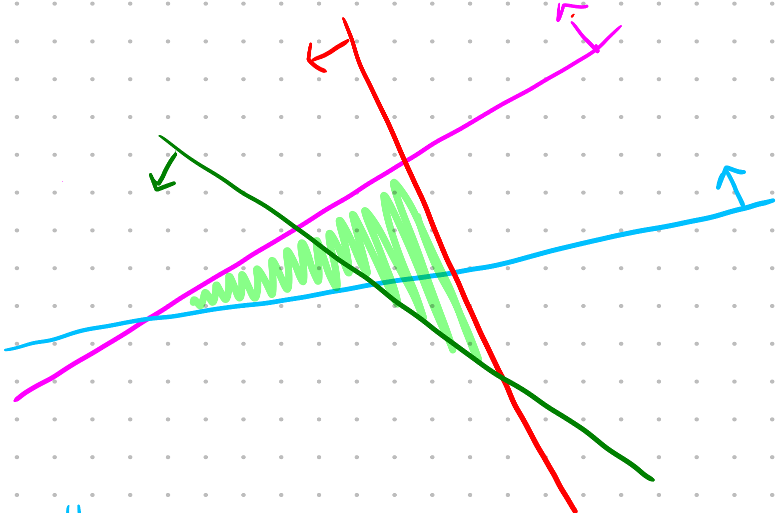
Grasstopes

Lizzie Pratt

With Yelena Mandelshtam & Dmitrii Pavlov



Slides: lizziepratt.com

The Grassmannian $Gr_{\mathbb{R}}(k,n)$

- Parameterizes k-subspaces in $\mathbb{R}^n$ $\left(= (k-1)\text{-planes in } \mathbb{P}_{\mathbb{R}}^{n-1}\right)$

- $Gr(k,n) = \mathrm{Mat}_{k\times n} \big/$ left multiplication by GL_k

eg:
- $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 \\ v_2 \end{bmatrix}$

- $\mathrm{rowspan}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathrm{rowspan}\begin{bmatrix} v_1 + v_2 \\ v_2 \end{bmatrix}$

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- Embed into $\mathbb{P}^{\binom{n}{k}-1}$ via $k\times k$ minors, called Plücker coordinates and denoted p_I, $I \in \binom{[n]}{k}$

eg: $\begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix} \rightsquigarrow [1 : c : -a : d : -b : ad-bc] \in \mathbb{P}^5$

I: 12 13 23 14 24 34

- Plücker relations: $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0.$

The Amplituhedron $\mathcal{A}(n,k,m)$

- The positive Grassmannian $Gr^{\geq 0}(k,n)$ is $\{V \in Gr(k,n) : P_I(V) \geq 0\}$

- The Amplituhedron $\mathcal{A}(n,k,m)$ is the image

$$
\begin{aligned}
\tilde{Z} : \; Gr^{\geq 0}(k,n) &\longrightarrow Gr(k, k+m) \\
\operatorname{span}\{v_1, \dots, v_k\} &\longmapsto \operatorname{span}\{Zv_1, \dots, Zv_k\} \\
A &\longmapsto AZ
\end{aligned}
$$

with Z totally positive $n \times (k+m)$

$$
\begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}
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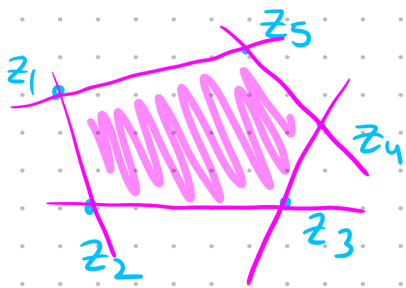
eg: $k=1$. Then

$$\tilde{Z} : (\mathbb{P}^{n-1})^{\geq 0} \longrightarrow (\mathbb{P}^{k+m-1})$$

$$[a_1 : \dots : a_n] \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \sum a_i z_i \ , \ a_i \geq 0.$$



$\Rightarrow$ im $\tilde{Z}$ is the cyclic polytope $C_n(Z)$

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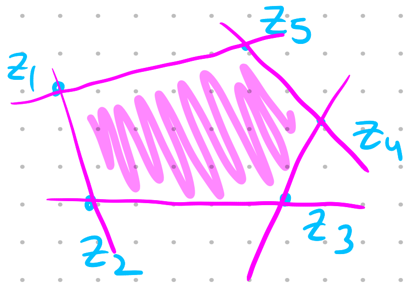

$\Rightarrow$ im $\tilde{Z}$ is the cyclic polytope $C_n(Z)$

Grasstopes

- A **Grasstope** $G_{n,k,m}$ is the image of

$$\tilde{Z}: Gr^{\geq 0}(k,n) \dashrightarrow Gr(k, k+m)$$

$$\text{span}\{v_1, \ldots, v_k\} \longmapsto \text{span}\{Zv_1, \ldots, Zv_k\}$$

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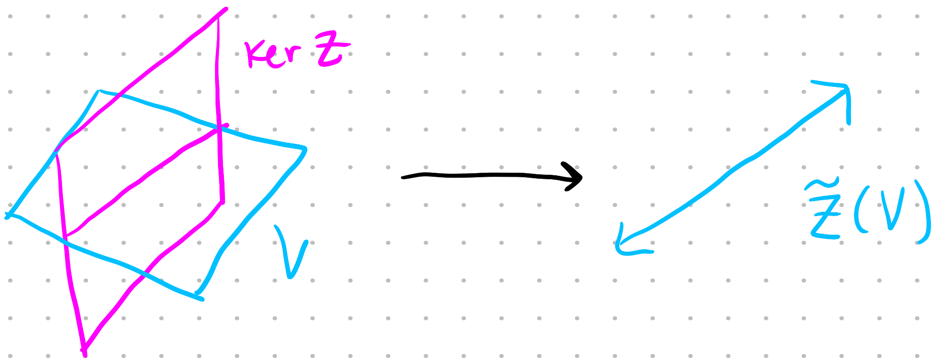
Could drop in dimension! >:(



Obs: $\tilde{Z}$ is not defined on (has base locus) the set

$$B(\tilde{Z}) = \{V \in Gr^{\geq 0}(k,n) : \operatorname{rk}(V \cap \ker Z) \geq 1\}$$

Sign variation

- The sign variation
  - $var(v)$ is the # of sign changes
  - $\overline{var}(v)$ is the # sign changes if each 0 is changed to maximize var.

eg: $var(1\ {-1}\ 2\ 1) = 2$

$\overline{var}(1\ 0\ 1\ 1\ 0) = 3$

$\overline{var}(+\ 0\ -) = 1$

Hyperplane Arrangements & $m=1$

$$\tilde{Z} : Gr^{\geq 0}(k,n) \longrightarrow Gr(k,k+1) \simeq \mathbb{P}^k .$$

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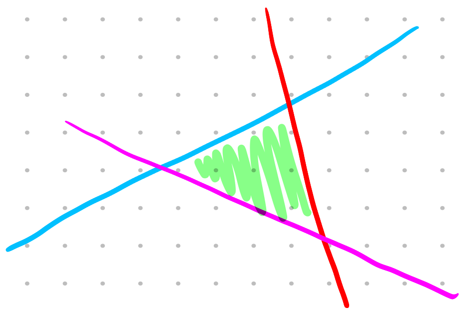
Thm: (Karp-Williams, 2019)

1) The $m=1$ amplituhedron consists of the closure of the bounded regions of the hyperplanes corresponding to rows of Z

eg:



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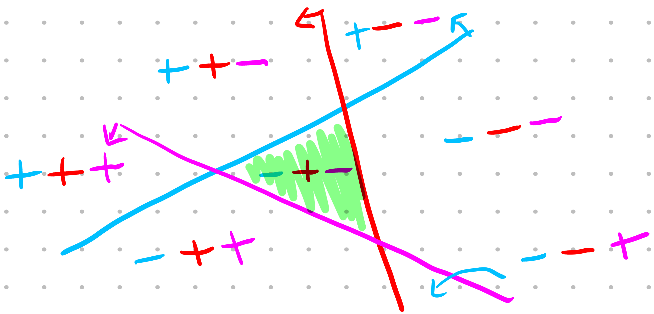
Thm: (Karp-Williams, 2019)

1) The $m=1$ amplituhedron consists of the closure of the bounded regions of the hyperplanes corresponding to rows of Z

2) The bounded regions are exactly the ones whose sign vectors σ have $\overline{\text{var}}(\sigma) \geq k$ with respect to the orientation of the hyperplanes

eg:



Projective Geometry & Duality

Q: Given a hyperplane $Y \in Gr(2,3)$ in Plücker coords, how to compute its equation in $\mathbb{R}^3$?

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eg: $x \in Y \iff \det\begin{bmatrix} x_1 & x_2 & x_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = 0 \iff p_{23}x_1 - p_{13}x_2 + p_{12}x_3 = 0.$

$Y = \mathrm{span}(v, w)$

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$Y = \mathrm{span}(v, w)$

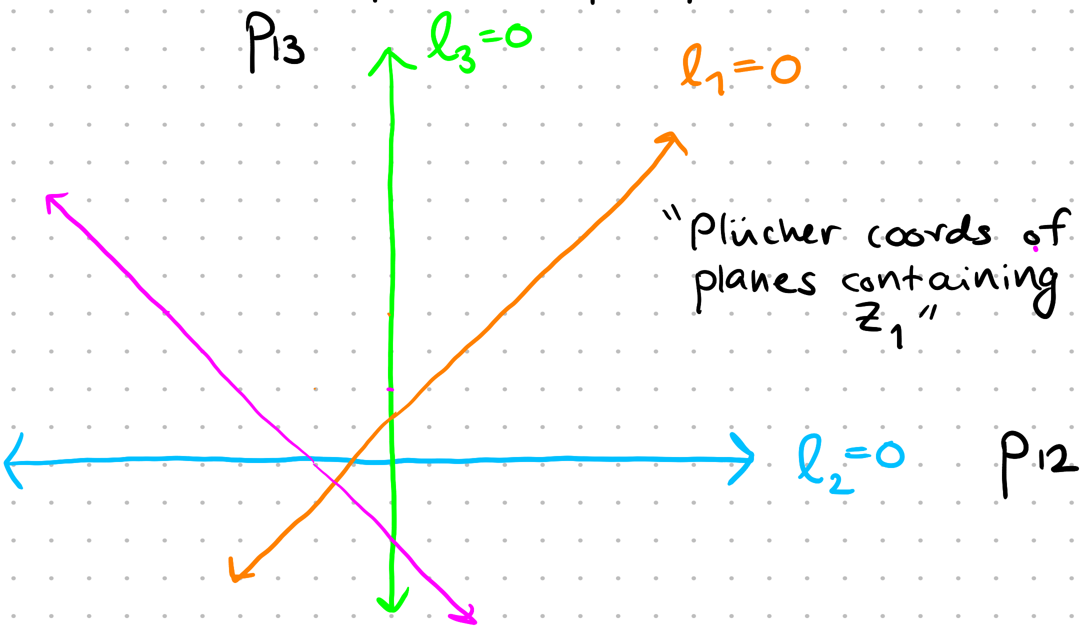
A: $x \in Y \iff \underbrace{\sum (-1)^{j+1} p_{I \setminus j} x_j}_{\langle Y, X \rangle : \mathbb{P}^k \times \mathbb{P}^k \longrightarrow \mathbb{R}.} = 0.$ bilinear form

Def: $l_i(Y) := \langle Y, Z_i \rangle = \langle Y i \rangle$ are called the Twistor coordinates of Y with respect to Z.

Example

$$Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix} \qquad \ell_i(y) = \begin{cases} p_{23}(y) & 1 \\ -p_{13}(y) & 2 \\ p_{12}(y) & 3 \\ (2p_{23}+2p_{13}+3p_{12})(y) & 4 \end{cases}$$

In the affine chart $p_{23} = 1 - p_{12} - p_{13}$:

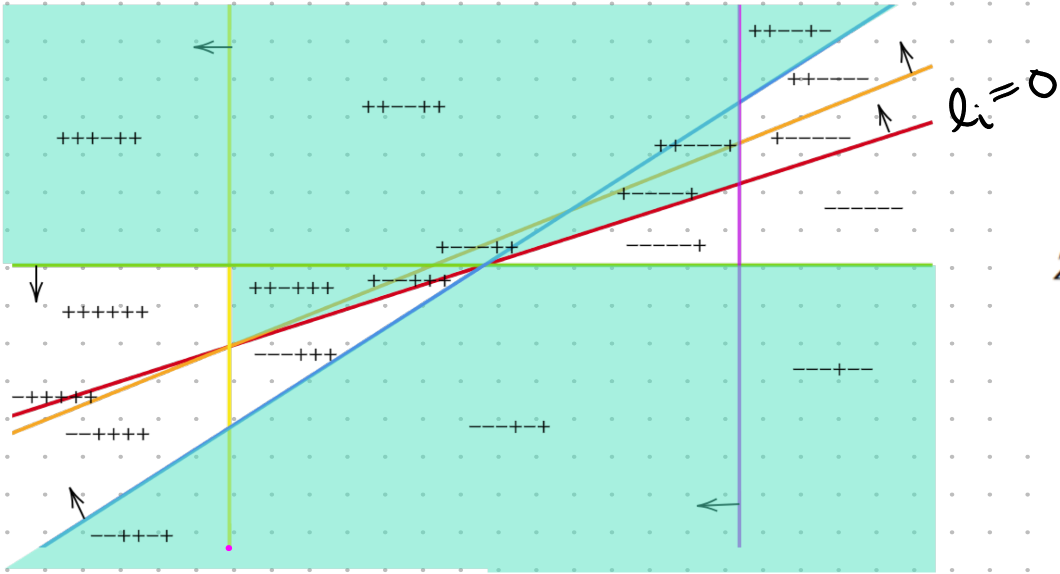


Main Result

Thm (Mandelshtam-Pavlov-P. 2023)

For Z with $B(\tilde{Z}) = \{0\}$, the Grasstope $G_{n,k,1}$ consists of all regions whose sign vectors σ have $\overline{\mathrm{Var}}(\sigma) \geq k$ with respect to the hyperplanes $\{l_i = 0\}$.

eg:



$$Z = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 3 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Topology

- The $m=1$ Amplituhedron: Closed, Connected, Contractible

- $m=1$ Grasstopes:*

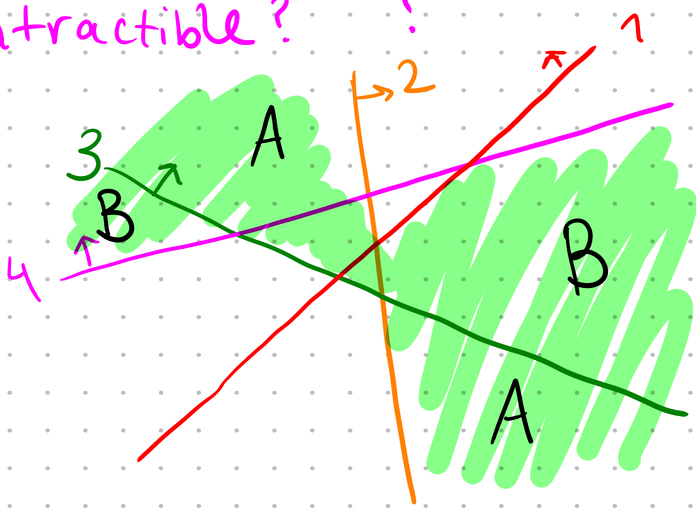
  > Closed? ✓

  > Connected? ✓

  > Contractible? ?

* for Z defined on $Gr^{\geq 0}(k,n)$

eg:

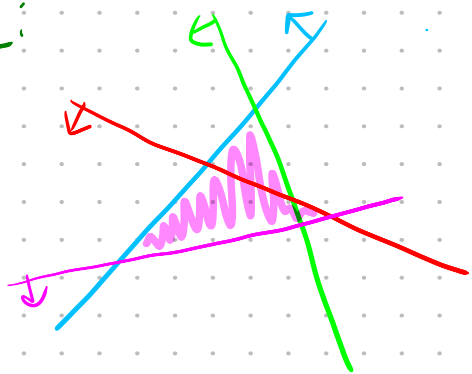


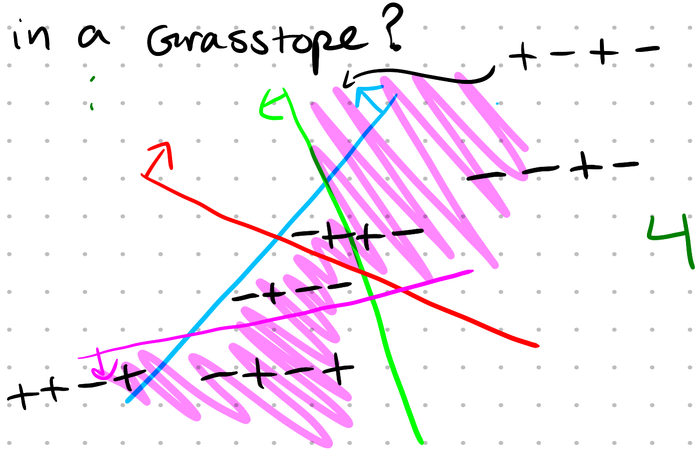
Not well-defined on $Gr^{\geq 0}(k,n)$

Extremal Counts

Q: How many regions are in a Grasstope?

eg:





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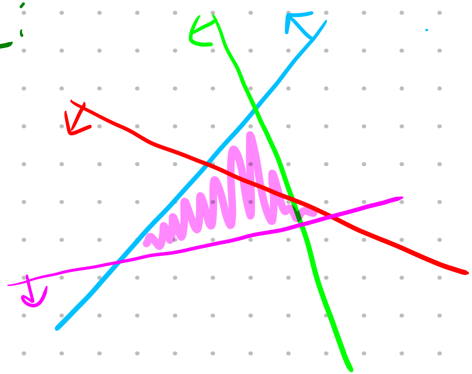

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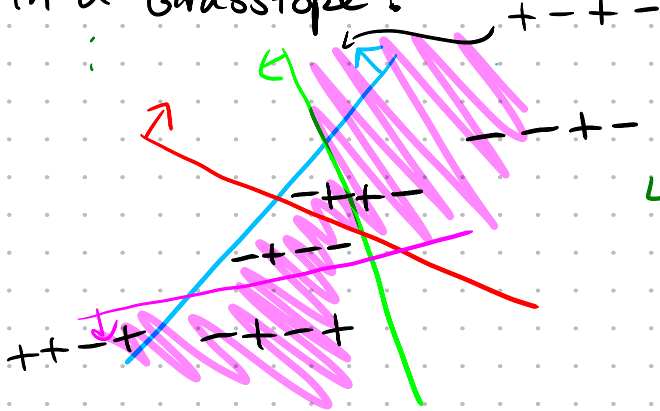


4

Observe

- # sign vectors v w/ $\mathrm{var}(v) \geq 2$: 4

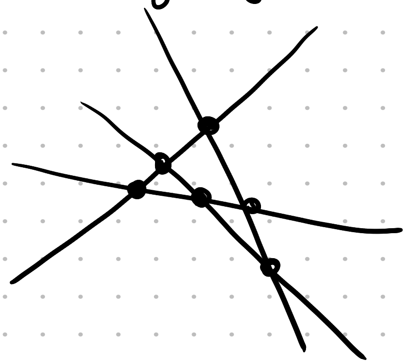
  $+-++,\ +--+,\ ++-+,\ +-+-$
- # of regions: 7
- Total # of sign vectors: 8

  $\Rightarrow$ set-theoretically, could get 3 or 4 regions

Extremal Counts, cont

Q: How many regions are in a projective hyperplane arrangement?

eg:



n lines

Extremal Counts, cont

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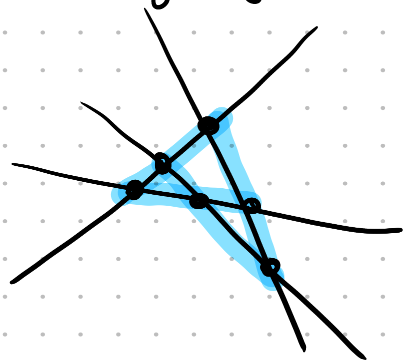

n lines

- $V = \binom{n-1}{2}$ 6
- $E = \frac{n(n-2)}{2}$ 4

$V - E + F = 2$

$\#\text{bounded} = F - 1$

- # bounded: $\binom{n-1}{2}$
- # unbounded: n

Extremal counts, cont

Thm (Zaslavsky).

The number of regions of an affine arrangement of n lines in $\mathbb{R}^k$ is

- $r(n) = 1 + n + \binom{n}{2} + \cdots + \binom{n}{k}$
- $b(n) = \binom{n-1}{k}$

Cor: The number of regions in a projective arrangement of n lines in $\mathbb{P}^k$ is $b(n) + \frac{r(n)-b(n)}{2} = r_{proj}(n)$

Observe:

- $\underbrace{\text{\# sign vectors with } var \geq k}_{:= \gamma(k,n)}$ is an upper bound
- $r_{proj} - \underbrace{\text{\# sign vectors with } var < k}_{:= \beta(k,n)}$ is a lower bound

Q: Are the bounds actually attained?

Lower Bound

Data: finschi.com

$k,\, n$	Minimal	Maximal	$r(\mathcal{P})$	$\beta(k,n)$	$\gamma(k,n)$
2, 6	10	16	16	6	26
2, 7	15	22	22	7	57
3, 5	4	5	15	11	5
3, 6	10	16	26	16	16
4, 6	5	6	31	26	6
4, 7	15	22	57	42	22

Table 1: Minimal and maximal possible number of regions in a Grasstope.

Lower bound of $r(P) - \beta(k,n)$ is attained by the amplituhedron.

Upper Bound

Data: finschi.com

k, n	Minimal	Maximal	$r(\mathcal{P})$	$\beta(k,n)$	$\gamma(k,n)$
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3, 6	10	16	26	16	16
4, 6	5	6	31	26	6
4, 7	15	22	57	42	22

Table 1: Minimal and maximal possible number of regions in a Grasstope.

The upper bound is attained!

Bounds

Data: finschi.com

```
++++++++++++++++++++++++-++++------: (20, 38)
+++++++++++++++++++++++++++-------+: (20, 39)
++++++++++++++++++++++++++++--+--++: (20, 38)
++++++++++++++++++++++++++++-++---+: (20, 39)
++++++++++++++++++++++++++++-++--++: (20, 39)
++++++++++++++++++++++++++++++-----: (20, 40)
++++++++++++++++++++++++++++++-+---: (20, 39)
++++++++++++++++++++++++++++++++---: (20, 40)
+++++++++++++++++++++++++++++++++--: (20, 40)
++++++++++++++++++++++++++++++++++-: (20, 41)
+++++++++++++++++++++++++++++++++++: (20, 42)
```

Amplituhedron

eg: $k=3,\ n=7$.

Upper Bound

Data: finschi.com

Q: what about the amplituhedron?

k, n	Maximal	$r(\mathcal{P})$	$\gamma(k,n)$
3, 7	42	42	42
3, 8	64	64	99
4, 8	64	99	64
5, 8	29	120	29
2, 9	37	37	247
3, 9	93	93	219
4, 9	163	163	163

Upper Bound

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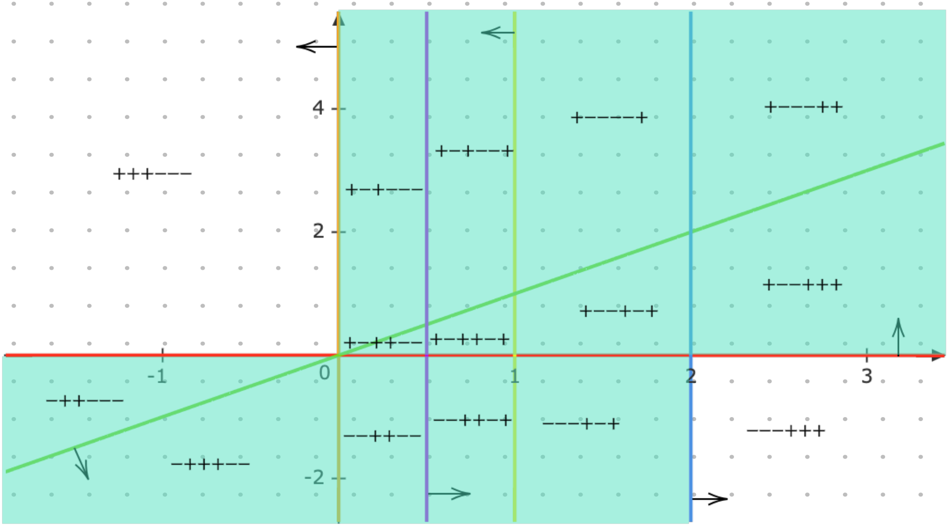
A: Attains upper bounds!

Thank You!



Code: mathrepo.mis.mpg.de/Grasstopes

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For Z with $B(\tilde{Z}) = \{0\}$, the Grasstope $G_{n,k,1}$ consists of all regions whose sign vectors σ have $\overline{\mathrm{var}}(\sigma) \geq k$ with respect to the hyperplanes $\{l_i = 0\}$.

Proof

Setup:

$$\begin{aligned} \mathbb{P}^k &\longrightarrow \mathbb{P}^{n-1} \\ x &\longmapsto [l_1(x) : \cdots : l_n(x)] \end{aligned}$$

For $u \in \mathbb{P}^{n-1}$,

$H_u = \{v : u \cdot v = 0\}$.

Claims

① H_u contains a positive k-dim subspace

$\iff \overline{\mathrm{var}}(u) \geq k$ (Gantmaher-Krein 1950)

② H_u contains a positive k-dim subspace

$\iff X \in \mathrm{im}\, \tilde{Z}$ where $u = [l_1(x) : \cdots : l_n(x)]$

Proof

Setup:

$$\mathbb{P}^k \longrightarrow \mathbb{P}^{n-1}.$$
$$X \longmapsto \langle X, z_i \rangle_i$$

Claim

H_u contains a positive k-dim subspace $\Longleftrightarrow$ $X \in \operatorname{im} \tilde{Z}$.

($\Leftarrow$) Suppose $X = pl(\tilde{Z}(A)) \in \mathbb{P}^k$.

Then $\sum_j \ell(X, z_j) A_{ij} = \ell(X, \underbrace{\sum_j A_{ij} z_j}_{= A_{i\cdot} Z \text{ is a row of } AZ}) = 0$.

$\Rightarrow$ H_u contains A.

($\Rightarrow$) Suppose H_u contains A.

- Let $v \in \ker Z$. Then $\sum \ell(x, z_i) v_i = \ell(x, \underbrace{\sum v_i z_i}_{=0}) = 0$.

  $\Rightarrow$ H_u contains $\ker Z$.

- Since $\ker Z \cap A = \emptyset$, $H_u = \ker Z \oplus A$.

  $\Rightarrow$ defines X uniquely, and $X = pl(\tilde{Z}(A))$ works.